Accuracy of Deterministic Extended-Path Solution Methods for Dynamic Stochastic Optimization Problems in Macroeconomics

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November 11, 2009

Abstract

The deterministic extended-path method for solving dynamic stochastic optimization problems approximates conditional expectations instead of approximating a model's complex non-linear dynamics. We show that this straightforward approach provides similar accuracy to the best results reported for alternative methods, and gives uniform performance across the entire state space. Our implementation requires roughly 4 fold more computer time than Galerkin projection, but the method has offsetting simplicity and generality that make it an attractive choice.

Keywords: *Dynamic stochastic equilibrium, conditional expectations, computational optimization, extended path, non-linear solutions.*

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1 Introduction

Relatively large Euler-equation errors have been reported for methods of solving dynamic stochastic general equilibrium macroeconomic models which employ linear and log-linear approximations (Aruoba et. al, 2006). The inaccuracies introduced by linearization can be severe enough to call the use of these simple and fast methods into question. Non-linear projection methods on the other hand, exhibit much better accuracy but can be difficult to implement (Heer and Mauβner, 2008) requiring decisions over approximation, integration, projection, and equation-solving procedures (Judd, 1998, pp. 606). This paper presents accuracy results for an alternative non-linear procedure, which largely limits the decisions facing a researcher to equation solving.

The deterministic "extended-path" (EP) method of Fair and Taylor (1983) essentially trades off errors due to linearization or other functional approx-

imations with expectational errors implied by a violation of Jensen's inequality. Early results on the accuracy of the EP method (Gagnon (1990), and Taylor and Uhlig (1989)) were promising but limited to qualitative comparisons with solution values from other methods. Here we report Euler-equation errors as computed in Judd (1992). We show that EP ranks as good or better than the most accurate methods reported by Aruoba et al. (2006)[1]. In addition, deterministic EP generates very uniform accuracy across the state-space and can be implemented without prohibitively long running times. The relative performance of deterministic EP does deteriorate for models with extreme non-linearities and extremely volatile stochastic processes, but it remains similar to projection methods and superior to log-linearization.

Section 2 below outlines the implementation of deterministic EP in this paper and discusses some details. Section 3 presents our accuracy results. Section 4 is a concluding discussion.

2 Deterministic EP.

2.1 An example model.

As in Aruoba et al. (2006) we employ the basic neoclassical workhorse model from the macroeconomics literature. The objective is the maximization of the expected life-time utility of a single representative household according to;

$$U = E\{\sum_{t=1}^{\infty} \beta^t u(C_t, (1 - l_t))\}. \tag{1}$$

C_t is period-t consumption, and l_t is labour supplied from a unit endowment per period. $\beta < 1$ is the household's subjective rate of time preference. The period utility function over consumption and leisure and is parameterized as:

$$u(C_t, l_t) = \tfrac{1}{1-\tau}(C_t^{\varepsilon}(1 - l_t)^{(1-\varepsilon)})^{(1-\tau)} \tag{2}$$

$$\varepsilon \in (0, 1),\ 0 \leq \tau \neq 1$$

$$u(C_t, l_t) = ln(C_t^{\varepsilon}(1 - l_t)^{(1-\varepsilon)}),\ \tau = 1.$$

Consumption choice is bounded each period by the aggregate resource constraint,

$$Y_t = K_{t+1} - (1 - \delta)K_t + C_t, \tag{3}$$

where Y_t is available output. $K_{t+1} - (1 - \delta)K_t$ is capital investment net of depreciation (δ). This determines the evolution of the state variable K_t, the aggregate capital stock, which in turn is combined with the labour supply choice in each period to produce output according to,

$$Y_t = A_t S_t K_t^{\alpha} l_t^{1-\alpha}, \tag{4}$$

where $\alpha \in (0, 1)$ and $A_t S_t$ represents the state of productivity of the economy. $A_t = (1 + g)A_{t-1}$,

[1] In independent work Heer and Mauβner, 2008 report similar Euler equation errors for deterministic EP methods.

$g \geqslant 0$ allows for exogenous productivity growth. $S_t > 0$ is a stochastic variable with process,

$$\begin{aligned} S_t &= exp\{z_t - \sigma_\epsilon^2/2(1-\rho^2)\}, \quad (5) \\ z_t &= \rho z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.\, N(0, \sigma_\epsilon^2)\, \forall t. \end{aligned}$$

These assumptions imply that $E[S_t] = 1$. Further, the T-step forecast of this process conditional on z_{t-1} is given by,

$$E[log(S_{t+T}) \mid z_{t-1}\} = \rho^{(T+1)} z_{t-1} - \frac{\sigma_\epsilon^2}{2(1-\rho^2)}, \quad (6)$$

with conditional variance of,

$$Var[log(S_{t+T}) \mid z_{t-1}] = \sigma_\epsilon^2 \sum_{i=0}^{T} \rho^{2i}. \quad (7)$$

A solution to this problem is a set of decision rules which, given current state-variable values (K_t, and $A_t S_t$), yield the current choice-variable values (C_t and l_t). Such a solution is fully characterized by equation 3, equation 5, and the following two first-order conditions,

$$\begin{aligned} \frac{(C_t^\varepsilon (1-l_t)^{(1-\varepsilon)})^{1-\tau}}{C_t} &= \\ \beta E_t \left[\frac{(C_{t+1}^\varepsilon (1-l_{t+1})^{(1-\varepsilon)})^{1-\tau}}{C_{t+1}} \cdot \right. & \quad (8) \\ \left. (1 + \alpha A_{t+1} S_{t+1} K_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta) \right] & \end{aligned}$$

$$\frac{(1-\theta)}{\theta} \frac{C_t}{1-l_t} = (1-\alpha) A_t S_t K_t^\alpha l_t^{-\alpha} \quad (9)$$

Equation 8 is the intertemporal Euler equation relating current and future marginal utilities. Equation 9 is a static first-order condition relating the marginal utilities of consumption and leisure in each period.

2.2 The Solution Method.

Closed-form analytical solutions for the decision rules of this model do not exist. Rather than solving directly for approximate closed-form decision rules, deterministic EP indirectly characterizes these decision rules by generating time-series that satisfy the model equations.

Consider first a deterministic case of the model. We have then $N = 4$ coupled first-order difference equations describing time-paths of the model's two choice and two state-variables, $\{l_t, C_t, K_t, A_t S_t\}_{t=0}^{t=\infty}$. Solution methods for such systems are readily available[2]. Central to these methods is the choice of how to effectively "close" the open system of difference equations. We employ a "two-point boundary" problem approach that closes the system by imposing the initial conditions given by the model's state variables *and* imposing suitable "terminal" conditions on the model's choice variables at some distant horizon T. This results in a closed system of $N \times (T+1)$ equations that can be efficiently solved up to machine precision. Long-run steady-states or fixed-point variable values are

[2] See Press et al. 1992, Chpt. 17 for a comprehensive discussion and related algorithms.

natural candidates for terminal conditions. Given a sufficiently large T (eg. such that $\beta^T \simeq 0$), this method eliminates two layers of iteration (type II and III) relative to the "shooting" type procedure outlined by Fair and Taylor (1983).

Of course, in the stochastic environment we have the added difficulty of computing the conditional expectation on the right side of equation 8. The deterministic EP method simply takes the conditional expectation inside the non-linear functions of this equation and solves at the expected value of the vector of future shocks conditional on current realizations. For example, conditional on the current technology, $A_0 S_0$, equations 6 and 7 can be used to calculate the expected values of future technology levels, $E_{t=0}\left\{A_t S_t\right\}_{t=1}^{t=T}$. Substituting these into the right side of equation 8 yields a deterministic system which is readily solved employing the methods described above. The resulting time-series contains solutions for the current periods decision variables (C_0 and l_0) conditional on the expectations assumed and on the current state. This solution also implies the next periods initial condition, K_1, which together with another draw from the stochastic process provides the starting point for finding the next period's decision variables. Iterating on this process q times yields a q-length time-series and the whole procedure can be replicated for an arbitrary number of vectors of realizations.

3 Solution Accuracy.

3.1 Euler Equation Errors.

For convenience we express the intertemporal Euler equation 8 as,

$$0 = \Gamma(C_t, l_t) - \int_{-\infty}^{\infty} \Omega(C_{t+1}, l_{t+1}, K_{t+1}, \epsilon_{t+1}) f(\epsilon_{t+1}) d\epsilon_{t+1}, \quad (10)$$

where $\Gamma(\bullet)$, and $\Omega(\bullet)$ are used to simplify notation, and $f(\epsilon_{t+1})$ denotes the distribution function for the stochastic component, ϵ_{t+1}, underlying the period-t+1 technological state. Along a solution path this equation should hold exactly. Deterministic EP solutions are only approximations, however, derived by substituting for the above integral with $\Omega(\bullet)$ evaluated at the expected value of ϵ_{t+1}, that is, $\Omega(C_{t+1}, l_{t+1}, K_{t+1}, 0)$. When evaluated at these approximate solutions, deviations of the right-hand side from zero will arise. As discussed in Judd (1992) and Aruoba et al. (2006) such deviations are a good measure of solution accuracy.

Because an exact solution for the integral in equation 10 is unknown, following Judd (1992) we approximate it with an 8-point, fifteenth-order accurate, Gauss-Hermite quadrature rule. Transforming $\epsilon_{t+1} \sim N(0, \sigma_\epsilon^2)$, as, $x_{t+1} \equiv \epsilon_{t+1}/\sigma_\epsilon\sqrt{2} \sim N(0,1)$, the integral can be expressed as,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Omega(C_{t+1}, l_{t+1}, K_{t+1}, x_{t+1}) exp(-x_{t+1}^2) dx_{t+1},$$

and approximated by the finite summation,

$$\frac{1}{\sqrt{\pi}} \sum_{i=1}^{8} \Omega(C_{t+1}, l_{t+1}, K_{t+1}, x_i) w_i, \qquad (11)$$

where x_i, and w_i are Guass-Hermite abscissas and weights.

With this integral approximation in hand the procedure for deriving Euler-equation errors is straightforward. Given the current-period state variables, K_t and ϵ_t, deterministic EP is used to obtain solutions for the current-period decision variables, $\{\hat{C}_t, \hat{l}_t\}$. These values imply $\hat{K}_{t+1}$ which is then coupled with each of the $\{x_i\}_{i=1}^{i=8}$ abscissa values in succession to form 8 possible t+1-period states for which 8 corresponding sets of t+1-period decision-variable solutions, $\{\hat{C}_{i,t+1}, \hat{l}_{i,t+1}\}_{i=1}^{i=8}$ are obtained, again by deterministic EP. The Euler-equation error associated with the current-period decision-variable solution can then be calculated as,

$$Eerror_t = \Gamma(\hat{C}_t, \hat{l}_t) - \frac{1}{\sqrt{\pi}} \sum_{i=1}^{8} \Omega(\hat{C}_{i,t+1}, \hat{l}_{i,t+1}, \hat{K}_{t+1}, x_i) w_i.$$

Lastly, these errors are expressed as percentages of the current-period consumption variable to provide an intuitive numeraire.

$$Cerror_t = \left[\left(\Gamma(\hat{C}_t, \hat{l}_t) - Eerror_t \right) \cdot (1 - \hat{l}_t)^{(\varepsilon-1)(1-\tau)} \right]^{\frac{1}{\varepsilon(1-\tau)-1}} \qquad (12)$$

3.2 Numerical Results.

For comparability with the results of Aruoba et al. (2006) we employ their benchmark parameterization of this model. We also examine their alternative parameterizations to guage the effects of highly non-linear and/or high variance specifications. Parameter values are reported in Tables 1 and 2.

Table 1: Benchmark Parameters

β	τ	ε	α	δ	ρ	σ_ϵ	g
.9896	2.0	.357	0.4	.0196	.95	.007	0

Table 2: Alternative Parameterizations

Case	$\sigma_\epsilon = 0.007$	$\sigma_\epsilon = 0.035$
$\tau = 2$	Benchmark	Intermediate 3
$\tau = 10$	Intermediate 1	Intermediate 4
$\tau = 50$	Intermediate 2	Extreme

A solution path was computed for each point on a grid of initial-values for the state-variables; $[K_t, \epsilon_t] \subseteq [0.7K_{ss}, 1.3K_{ss}] \times [5\sigma_\epsilon, -5\sigma_\epsilon]$, where $K_{ss} = 23.14084$ is the fixed-point or steady-state of K_t. This covers virtually 100% of the model state-space. With $\sigma_\epsilon = 0.007$, 528 equally-spaced points were used. With $\sigma_\epsilon = 0.035$ the step-size for ϵ_t was halfed yielding 860 points.

Table 3 reports in base 10 logs the maximum, the average, and sample confidence intervals around the average, of the Euler equation errors calculated from the first two periods of each solution path. For comparison purposes the table also reports results from Aruoba et al. (2006). A path length of T=200 was used in all cases. Increasing this to T=300 in

Table 3: Euler Errors as %Cons. ($log_{10}(abs.)$)

Det. EP	Max.	Avg.	Avg.$\pm 3s.e.$
Benchmark	-5.1654	-5.2648	(-5.138,-5.444)
Inter. 1	-4.7936	-4.9031	(-4.765,-5.108)
Inter. 2	-4.2337	-4.3864	(-4.254,-4.579)
Inter. 3	-3.6992	-3.8654	(-3.677,-4.206)
Inter. 4	-3.3431	-3.5064	(-3.325,-3.823)
Extreme	-2.8650	-3.0150	(-2.814,-3.402)

Value Function*	Max.	Avg.
Benchmark	-4.4343	-5.6498
Extreme	-4.015	-4.4949

Chebyshev*	Max.	Avg.
Benchmark	-3.3281	-5.4330
Extreme	-2.5269	-4.6578

Log-Linear*	Max.	Avg.
Benchmark	-2.2002	-4.2002
Extreme	-1.4315	-2.6131

* Value-Function iteration, Spectral method with Chebyshev polynomials, and Log-Linear approximation results as reported in Aruoba et al. (2006).

the benchmark case improved the average error by only -2.85e-06.

The errors resulting from the EP method are small. The *maximum* Euler error for the benchmark case implies a $1 optimization mistake for each $178,000. As expected this performance deteriorates as the non-linearities and stochastic volatility in the model increase. For the "extreme" calibration, *maximum* mistakes are on the order of $1 per $600. This, however, is no worse than the maximum Euler error reported by Aruoba et al. (2006) for the spectral projection method (Chebyshev) and is significantly better than the log-linear approximation.

There is also relatively little variation in the accuracy of the EP method across the state-space as indicated by the reported confidence intervals. This is because the method does not rely on discretization or other state-space restrictions nor on approximations relative to a fixed point (steady-state). Thus, while projection methods for example are more accurate on average, particularly under the extreme calibration, the EP method provides uniformly acceptable accuracy across the state-space and far from the steady-state. In anycase the EP method continues to outperform log-linearization.

3.3 Robustness.

The first three rows of Table 4 explore alternative calibrations from Auroba et al. (2006). As expected, the EP method delivers Euler equation errors near machine zero in the deterministic case. Log-utility and higher discounting (low β) yield errors comparable to the benchmark.

The fourth row of Table 4 shows accuracy close to machine zero for a near-linear parameterization where certainty equivalence applies. The fifth and sixth rows present results for important alternative specifications; first where the exogenous productivity factor, S_t, is assumed to follow a random walk ($i.e.\ \rho \to 1$);

$$ln(S_t) = ln(S_{t-1}) + \xi_{St}, \quad \xi_{St} \sim N(0, \sigma_\xi^2), \quad (13)$$

and second allowing for exogenous growth of 4% per annum ($g = 0.02$ in $A_t = (1+g)A_{t-1}$). In both cases, the EP method continues to deliver excellent accuracy comparable with the benchmark.

Table 4: Robustness experiments

Model Variation	Max	Avg
deterministic ($\sigma_\varepsilon^2 = 0$)	-14.386	-15.025
log utility ($\tau = 1$)	-5.2198	-5.3132
high discount ($\beta = 0.98$)	-5.0383	-5.1257
linear ($\tau = 1, \delta = 1$)	-11.611	-11.665
random walk (eqtn:13)	-4.4808	-4.5035
growth ($g = 0.02$)	-5.173	-5.191

4 Discussion.

This paper shows that the deterministic extended-path method of Fair and Taylor (1983) solves dynamic stochastic general equilibrium models with similar accuracy to the best results reported for alternative solution methods. The relative performance of the method is sensitive to extreme non-linearities and extremely volatile stochastic processes. However, its accuracy remains superior to that of linearization methods, and it continues to yield very uniform results across the state space and for large intervals around the steady-state.

Both linearization methods and deterministic EP rely on the imposition of certainty equivalence to evaluate conditional expectations. Linearization modifies the original structure so that certainty equivalence holds, whereas deterministic EP ignores the violation of Jensen's inequality caused by taking the conditional expectation inside the non-linear functions of the original structure. In either case the approximation error depends on the degree of nonlinearity of the original model. For the class of models documented here, the similar accuracy of deterministic EP and spectral projection methods argues that violating Jensen's inequality is of relatively minor consequence. The superiority of both of these methods to log-linearization places emphasis on retaining the nonlinearities of the original model structure.

A disadvantage of deterministic EP may be relatively high computing times. For the benchmark two state-variable model studied here, computing 500 simulations of 60 periods each (i.e. 30,000 points) takes *12 mins., 18 secs.*, on a 2.8 Mhz Pentium IV running Linux (kernal 2.63) as compared with *3 mins., 44 secs.* reported for Galerkin projection by Heer and Mauβner (2008). Thus, while not prohibitive, the use of deterministic EP for estimation purposes may be computationally extensive.

On the otherhand, programming and debugging time is limited mostly to the implementation of standard non-linear solution techniques applied directly to the model's first-order conditions in stationary form. As with all non-linear computations, good starting values are essential. The stationary path defined by the model's fixed-point, which typ-

ically corresponds to the terminal boundary conditions, readily gives such starting values to aid in debugging, and from which highly divergent solutions can easily be built up by iteration.

Since deterministic EP works straight from first-order conditions it is also easily applied to models with non-convexities, side-conditions, and inequality contraints. Lastly, the method does not suffer a "curse of dimensionality" making it potentially useful for systems with several state variables.

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