Favoritism in Contests: Head Starts and Handicaps*

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Abstract

We examine a contest, modelled as an all-pay auction, in which a strong and a weak contestant compete, and where a contestant may suffer from a *handicap* or benefit from a *head start*. The former reduces the contestant's score by a fixed percentage; the latter is an additive bonus. The two instruments affect the contest in significantly different ways. In particular, a handicap does not "cancel out" a head start. The effort maximizing combination of head starts and handicaps is then analyzed. In the benchmark model, it is generally profitable to give the weak contestant a head start. However, we identify a trade-off which implies that it may or may not be profitable to handicap the strong contestant. Indeed, the weak contestant may have a head start *and* a handicap. The trade-off is absent in a perturbed model, but there it is unambiguously the weak contestant who should be handicapped.

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1 Introduction

Writing a grant proposal can involve a significant investment of time and effort. However, depending on how the applicant and the application itself is evaluated, this investment is not always rewarded. Given several researchers compete for limited funds, even good applications are often unsuccessful.

In Canada, two criteria are used for evaluating an application for a Standard Research Grant submitted to the Social Sciences and Humanities Research Council (SSHRC). Specifically, "the score on the record of research achievement accounts for 60 per cent of the overall score, and the score on the program of research accounts for 40 per cent of the overall score".[1] The implication is that an individual with a good research record has a *head start*. That is, she will out-score a competing proposal by a less successful scholar, even if the proposed programs of research are of comparable quality.

However, an applicant will normally be considered as a "new scholar" if he obtained the Ph.D. within the last five years of the application deadline. In this case, the two components are weighted "such that either a 60/40 or 40/60 ratio will apply, depending on which will produce the more favorable overall score."

Now, compare an established scholar and a new, unproven, scholar. The established scholar has a *head start* due to her past research achievements. On the other hand, she is *handicapped* in the sense that pouring more effort into the proposal impacts her score less (the program of research accounts for only 40%) than a corresponding increase in effort by the new scholar would increase his score (since the program of research accounts for up to 60% for this applicant).[2] At first sight, it may appear that this design is self-contradictory; it is certainly unclear which scholar is favoured.

The topic of this paper is favoritism in contests in which contestants may be heterogeneous. Given the numerous instances in which contests are manipulated, it is surprising that favoritism has been subject to only limited formal study in the standard models of contests. The main objective of the paper is therefore to challenge some common intuitions in a formal model. The following results are among the main findings: (1) a contestant who is favoured does not necessarily "slack off"; he may respond by working harder, (2) it may be profitable to handicap the *weak* contestant,

[1] See http://www.sshrc.ca/web/apply/program_descriptions/standard_e.asp.

[2] We ignore the time element in this discussion, as well as in the model. In practice, there may be an incentive to devote the time and effort to building up the CV this year, in order to increase the chances in next year's competition. In other words, we consider the record of research achievement to be fixed when the decision to write a grant proposal is made. See Konrad (2002) for a two-stage model in which preliminary actions in the first stage affects the contest, held in stage two.

and (3) it may be more profitable still to combine various instruments in ways that appear self-contradictory; specifically, to simultaneously give a contestant a head start and a handicap. The emphasis in the paper is on the second and third point, concerning the profitability of favoritism.

To begin, we posit that a statement such as "it is profitable to favour the weak contestant" is too simplistic and possibly even false, even though it is intuitively compelling. Leveling the playing field may not be profitable.

First, there are many ways to favour a contestant, and it must be recognized that different ways of manipulating a contest may have very different consequences. Care should be taken to be precise about the form the favoritism takes; the devil is in the details. In this paper, we illustrate this point by distinguishing between two instruments, namely *head starts* and *handicaps*.[3] These instruments affect the contest in very different ways, meaning that one is not a substitute for the other. Giving a contestant a head start is *not* equivalent to handicapping his opponent, although both measures are to his advantage.

Second, while it *is* true in the benchmark model that it is profitable to favour the weak contestant – if it is done with the right instrument – the most surprising result is perhaps that using the wrong instrument to do so may backfire. In particular, it may, at least under some circumstances, be profitable to handicap the weak contestant in favour of the strong contestant. As a result, we will show that the contest design implemented by SSHRC, involving a head start counteracted by a handicap, may be very effective at increasing the average quality of the program of research.[4,5] In particular, this design may be profitable if a grant is more valuable early in a scholars' career, such that new scholars are believed to value a grant substantially more than established scholars.

More specifically, the contest is modelled as an *all-pay auction* in which bidders are privately informed about either their valuation of the prize or their ability (incomplete information). As explained in Section 5, the assumption of incomplete

[3]Head starts and handicaps are identity-dependent. Alternatively, the contest can be manipulated by imposing restrictive rules that apply to all contestants. There are several papers on caps in contests, i.e., an upper bound on how much effort or money the contestants can expend. Although the cap applies to all contestant, the strategic response may be different for weak and strong contestants. See Che and Gale (1998), Gavious et al. (2002), and Sahuguet (2006). Caps are briefly discussed in Section 5.

[4]Incidentally, this work is *not* funded by SSHRC.

[5]There are at least two reasons for why it may be in the funding agency's interest to increase the quality of the proposed program of research. First, it is likely that there is a positive correlation between how much effort goes into the proposal and how well the money are spent. Second, impressive and visionary applications may make it easier to justify the existence of the funding agency to the government. In fact, SSHRC may make successful applications public.

information is critically important. Bidders are ex ante heterogeneous, with one bidder perceived to be more likely to have high valuation or ability than another. Since we wish to emphasize the role of this heterogeneity in determining the optimal design features, we impose a few simplifying assumptions to facilitate the analysis. As in most of the existing literature on asymmetric auctions, we assume there are exactly two bidders, one "strong" and one "weak". This assumption is discussed in Section 5. We also assume that bidders are risk neutral and that the cost of bidding is linear in the bid. These assumptions permit the use of powerful arguments from mechanism design, and ultimately yields a rich set of results and insights.[6]

In the auctions that we examine, two design features imply that the bidders do not necessarily compete on even terms. A bidder has a *head start* if he would win the auction if both bidders bid zero. On the other hand, a bidder is *handicapped* if an increase in his bid has a smaller impact on how his participation is evaluated compared to an equal-sized increase in his competitor's bid.[7] Concretely, if bidder i bids b, his "score" is $s_i = a_i + r_i b$. The winner of the auction is the bidder with the highest score, not necessarily the bidder with the highest bid. Bidder 1 has a *head start* if $a_1 > a_2$, and is *handicapped* if $r_1 < r_2$.

The important observation is that a handicap has bearing on the marginal return of increasing the bid, while a head start does not. Consequently, the two instruments affect the auction in very different ways. Roughly speaking, a head start influences the decision to *participate* in the auction. In contrast, a handicap influences the *relative scores* of bidders who have decided to participate.

There are numerous examples of contests in which head starts and/or handicaps play important roles. Examples where they are imposed by a contest designer include sports (e.g. golf and horse racing), affirmative action, and uneven treatment of internal and external applicants for senior positions. They are sometimes exogenous and fixed components of the competition, as in R&D races between firms with different technologies and R&D procedures. Finally, they are sometimes created over time in dynamic contests, such as in a R&D race where one firm, perhaps by chance, is first to make a preliminary discovery that may function as a stepping stone.

The literature on favoritism in all-pay auctions is quite small. Konrad (2002) examines a two-bidder model with head starts and handicaps. However, he assumes

[6] A few papers, including Moldovanu and Sela (2001) and Gavious, Moldovanu, and Sela (2002), consider the possibility that the cost of bidding is non-linear in the bid. However, they assume that bidders are homogeneous. Clark and Riis (2000) assume bidding costs are non-linear and that bidders are heterogeneous, but specifically assume that types are drawn from (different) uniform distributions.

[7] In the SSHRC example, the "bid" is the program of research. A better program is costly to the bidder, since it requires more time and effort.

the value of the prize is known (complete information) and that it is the same for both bidders (homogeneous bidders). Moreover, Konrad (2002) assumes the head start and handicap are exogenously given, whereas we allow them to be determined by a third party. In an inspired paper, Siegel (2008) analyzes a very general model of contests that encompasses complete information, all-pay auctions with exogenous head starts and handicaps. In the current paper, however, we assume information is incomplete (see Section 5 for a discussion of the importance of this assumption). With incomplete information, there appears to be no previous papers dealing with head starts, and only two that examine handicaps. Feess et al (2008) assume the two bidders are homogeneous, while Clark and Riis (2000, henceforth C&R) allow the two bidders to be heterogenous, although they specifically assume types are drawn from uniform distributions. In their model, C&R find that it is profitable to handicap the strong bidder, and claim that the reason is that it "evens up" the contest.

However, it is demonstrably false that it is always the strong bidder who should be handicapped. In the benchmark model, where both bidders are interested in the prize with probability one, we observe that employing a handicap will necessarily involve a trade-off. In this regard, it is fruitful to think of the auction with and without a handicap as simply different *mechanisms*, thus allowing arguments from mechanism design to illuminate the problem. Without the handicap, the weak bidder tends to "overcompensate" for his weakness if his type is high. Consequently, he wins much too often in the all-pay auction compared to the revenue maximizing mechanism when his type is high. Handicapping the strong bidder only makes this worse.[8] On the other hand, the weak bidder bids very timidly if his type is low. Hence, he wins far less often in the all-pay auction compared to the revenue maximizing mechanism when his type is low. But, handicapping the strong bidder alleviates this problem. Thus, there is always a trade-off when a handicap to some bidder is introduced. In the C&R model, it happens to be the case that handicapping the strong bidder is optimal, on balance. However, it is easy to construct examples where it is the weak bidder who should be handicapped. In summary, it is not necessarily desirable to make the contest appear more even.

A head start affects the allocation only if bidders' types are low. As mentioned above, the standard all-pay auction has the drawback that the weak bidder does not win often enough when his type is low. Giving the weak bidder a head start is an obvious solution, precisely because it allows him to win more often when his type is low. Note that when a small head start is extended to the weak bidder, the positive effect of handicapping the strong bidder is diminished, while the negative effect is

[8]Notice also that handicapping the strong bidder may make it more likely that the auction is won by the bidder with the lowest valuation.

independent of the head start. Thus, when a head start is allowed, there is less reason to handicap the strong bidder. Indeed, in the C&R model, we find that if the asymmetry among bidders is sufficiently large, it is optimal to give a head start to the weak bidder, and then handicap him as well.

The existence of the aforementioned trade-off complicates the analysis. We therefore briefly consider a perturbation of the model where the trade-off is absent. In this model, the weak bidder is uninterested in the prize with positive probability. However, contingent on being interested, he is just as interested as the strong bidder. Here, the weak bidder *always* overcompensates when he is interested. Thus, it is *unambiguously* profitable to handicap the *weak* bidder. To use C&R's terminology, the contest is made *more uneven* in such cases. However, we point out that although the *environment* may appear more uneven, the *outcome* may in fact become more even once the strategic impact of the changing environment in taken into account.

The remainder of the paper is organized as follows. The benchmark model is introduced in Section 2, and equilibrium strategies are derived. Optimal head starts and handicaps are considered in Section 3. A perturbed model is examined in Section 4. Section 5 discusses the results and possible extensions. Section 6 concludes.

2 Model and Equilibrium

We model the contest as an all-pay auction. There are two bidders.[9] Each bidder is characterized by a privately known type which captures the value the bidder puts on winning the auction or the prize in the contest. Bidder i draws his type from the continuously differentiable distribution function F_i with support $[0, \overline{v}_i]$, $i = 1, 2$. Hence, F_i describes bidder j's beliefs about bidder i. We assume the distribution function has no mass points, although this assumption is relaxed in Section 4. The density, $f_i(v) = F_i'(v)$, is finite and bounded away from zero.

The two bidders are heterogeneous if $F_1 \neq F_2$. The common assumption in the auction literature is that the distributions can be ranked according to first order stochastic dominance, meaning that one bidder is more likely to have a low type.

Assumption A (Bidder 1 is weak, bidder 2 is strong): Assume that $\overline{v}_1 \leq \overline{v}_2$ and $F_1(v) > F_2(v)$ for all $v \in (0, \overline{v}_1)$.

We will maintain this assumption for expositional simplicity, such that we distinguish between weak and strong bidders. However, the formal results and insights

[9]See Parreiras and Rubinchik (2006) for an analysis of aspects of all-pay auctions with many heterogenous bidders.

rely only on the following more technical assumption, which we also impose.

Assumption B (No tangency at the end-points): Assume that $f_1(0) > f_2(0)$ and that either (i) $\overline{v}_1 < \overline{v}_2$ or (ii) $\overline{v}_1 = \overline{v}_2$ and $f_1(\overline{v}_1) < f_2(\overline{v}_2)$.

Assumption B is neither stronger nor weaker than Assumption A. Assumption B is silent on how the distributions compare in the interior (it does not rule out that F_1 and F_2 cross), but it is more particular regarding how the distributions compare at the end-points.

In a standard all-pay auction bidder i must decide whether to participate in the auction, and, if so, which non-negative bid, b_i, to submit, $i = 1, 2$. The bidder with the highest bid would then be the winner. Here, however, we assume bidders receive differential treatment. It is convenient to think of bidder i as accumulating a "score", s_i. If bidder i bids b, his score is $s_i = a_i + r_i b$, where $a_i \geq 0$, $r_i > 0$. The winner is then the bidder with the highest score.[10] Hence, bidder i must decide whether to participate, and, if so, which score to aim for.

We say that bidder 1 has a *head start* if $a \equiv a_1 - a_2 > 0$, in which case he wins the auction if both bidders bid zero. Moreover, bidder 1 is *handicapped* if $r \equiv r_1/r_2 < 1$, while bidder 2 is handicapped if $r > 1$. The handicapped bidder's score responds less to an increase in the bid than is the case for the other bidder.

Turning to payoffs, we assume that bidders are risk neutral and that the true cost of a bid of b is in fact b. These assumptions are standard in the auction literature, but a more general treatment would allow for risk aversion and costs that are non-linear in the bid. The cost of obtaining a score of $s \geq a_i$ for bidder i is

$$c_i(s) = \frac{s - a_i}{r_i}. \tag{1}$$

2.1 Equilibrium allocation

We let $\varphi_i(s)$ denote bidder i's inverse strategy, i.e. bidder i scores s in equilibrium if his type is φ_i, $i = 1, 2$. Assuming that strategies are strictly increasing in type among the set of types that participate in the auction, the probability that bidder i's score is at or below s is $F_i(\varphi_i(s))$.

Given the rival's strategy, bidder i with type v maximizes his expected payoff. That is, he solves the problem

$$\max_{s \geq a_i} vF_j(\varphi_j(s)) - c_i(s),$$

[10]If $a_i > a_j$ and $s_1 = s_2 = a_i$, we assume that bidder j wins. This assumption ensures the existence of an equilibrium. The tie-breaking rule is inconsequential in all other cases.

or, equivalently,

$$\max_{s \geq a_i} r_i v F_j(\varphi_j(s)) - s \tag{2}$$

where $j \neq i$ is bidder i's competitor. Other things being equal (in particular $\varphi_j(s)$), the size of r_i directly affects the *return* of scoring higher. Consequently, it should be no surprise that r_i will influence the *relative scores* of the two bidders, and thus the probabilities of winning. In contrast, given the assumptions of risk neutrality and linear costs, a_i is significant only in that it determines the lowest feasible and rational score. If a_i is very high, bidder j may decide to stay out if his type is low since a strictly positive bid cannot be rationalized if it leads to a score below a_i. However, among the types that do participate, a_i does not distort the allocation. In other words, a_i will have a *level* effect on the scores, and may force out some of bidder j's types.

Assuming, for now, that the solution is interior, the first order condition is

$$r_i v f_j(\varphi_j(s)) \frac{d\varphi_j(s)}{ds} = 1.$$

In equilibrium, bidder i with type v obtains a score of s, meaning that $v = \varphi_i(s)$, and the first order condition can be written as

$$\frac{d\varphi_j(s)}{ds} = \frac{1}{r_i \varphi_i(s) f_j(\varphi_j(s))}. \tag{3}$$

As in Amann and Leininger (1996), dividing the two first order condition yields

$$\frac{d\varphi_1(s)}{d\varphi_2(s)} = r \frac{\varphi_1(s) f_2(\varphi_2(s))}{\varphi_2(s) f_1(\varphi_1(s))}. \tag{4}$$

To continue, define $k(v)$ as the type of bidder 1 who obtains the same score as bidder 2 with type v. In other words, bidder 2 with type v wins if bidder 1's type is below $k(v)$. Since bidder 2 of type $\varphi_2(s) = v$ achieves the same score as bidder 1 with type $\varphi_1(s) = k(v)$, (4) can be rewritten as

$$\frac{dk(v)}{dv} = r \frac{k(v) f_2(v)}{v f_1(k(v))}. \tag{5}$$

To solve the differential equation we make use of the boundary condition that $k(\overline{v}_2) = \overline{v}_1$. That is, the two bidders must share a common maximal score. Otherwise, the bidder with the highest possible score could reduce his bid without reducing the probability that he wins. With this boundary condition, (5) is then sufficient to completely determine $k(v)$. In particular, $k(v)$ must satisfy

$$\int_{k(v)}^{\overline{v}_1} \frac{f_1(x)}{x} dx = r \int_{v}^{\overline{v}_2} \frac{f_2(x)}{x} dx. \tag{6}$$

For any $v > 0$, there exists some (unique) k which satisfies (6). The right hand side is finite for any $v > 0$, while the left hand side is 0 if $k = \overline{v}_1$ but increases and goes to infinity as k approaches 0 from above.

It is important to recognize that although $k(v)$ depends or r it is *independent* of a_1 and a_2. This confirms that a_1 and a_2 does not affect the *relative bids* of the bidders (among the active types), since k by definition reveals who bidder 2 ties with. However, we have not yet determined how the entry decision and the *level* of the bids is affected by changes in a_1 and a_2.[11] To do so, we assume for the sake of exposition that bidder 1 is the bidder with the head start, $a \geq 0$. As we will see in the next section, this is indeed profitable.

Since bidder 1 has a head start, bidder 2 may decide not to enter the auction at all. In particular, bidder 2 realizes that any score below a_1 will fail to win him the auction. Hence, bidder 2 either decides that it is too costly to participate and thus stays out if his type is sufficiently small, or he submits a bid of at least $c_2(a_1)$. Thus, no score below a_1 will be observed in the auction. It will never be profitable for bidder 2 to participate if $c_2(a_1) \geq \overline{v}_2$. In the following we therefore focus on the case where $c_2(a_1) \in [0, \overline{v}_2)$.

To find the critical type of bidder 2 who is indifferent between staying out of the auction and entering the auction with a score of a_1, we solve

$$vF_1(k(v)) - c_2(a_1) = 0. \tag{7}$$

Let the solution be denoted by v_2^c, and define $v_1^c \equiv k(v_2^c)$. Clearly, v_1^c and v_2^c depend directly on a and r_2. However, they also depend on r_1, since k depends on r_1.

We are now ready to outline the main properties of the equilibrium. In (v_2, v_1) space, Figure 1 depicts $v_1 = k(v_2)$ (as defined by (6)) as well as the level curve on which $v_2F_1(v_1)$ is constant and equal to $c_2(a_1)$. Note that the former is increasing, by (5) or (6), while the latter is decreasing. The intersection of the two satisfies (7) and thus defines v_1^c and v_2^c.

[11] In particular, k was derived under the assumption that the first order conditions are satisfied, i.e. that the solution is interior. However, this is not a valid assumption for low types when $a \neq 0$.

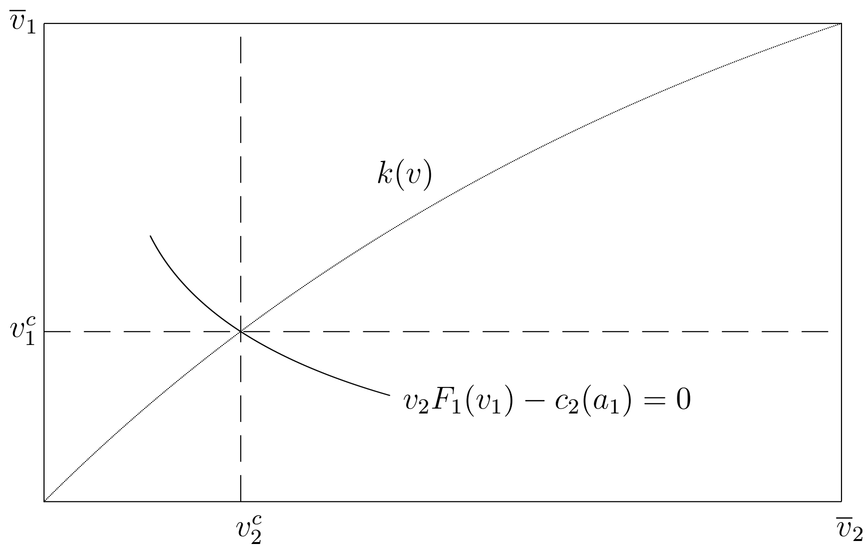

Figure 1: The equilibrium allocation. Bidder 2 wins below $k(v)$, to the right of v_2^c. Bidder 1 wins everywhere else.

In equilibrium, bidder 2 stays out of the auction if his type if strictly below v_2^c, and enters with a bid of $c_2(a_1)$ (score of a_1) if his valuation is precisely v_2^c. If his valuation is higher, he enters the auction and obtains a score equal to that obtained by bidder 1 with type $k(v)$. Bidder 1 enters the auction regardless of his type, but he submits a bid of zero, thereby obtaining a score of a_1, if his type is v_1^c, or below. If his type is higher he achieves a score to rival that of bidder 2 with type $k^{-1}(v)$.

As any point in Figure 1 represents a possible combination of types, it can be used to illustrate who wins the auction as a function of the bidders' types. The central point is (v_2^c, v_1^c). To the left of this point, $v_2 < v_2^c$, bidder 2 stays out of the auction, meaning that bidder 1 wins regardless of his type. To the south-east of (v_2^c, v_1^c), bidder 1 bids zero (scores a_1), while bidder 2's type is so high that he decides to be active in the auction (score above a_1). Hence, bidder 2 wins. Both bidders are active to the north-east of (v_2^c, v_1^c), but bidder 1 wins if the combination of types is above the $v_1 = k(v_2)$ curve, while bidder 2 wins below it.

Clearly, the allocation in the auction can be manipulated by manipulating the two curves in Figure 1. The "level curve" can be moved to the right by increasing a or decreasing r_2, while k can be made to move down by increasing r.

In Figure 1, an increase in a corresponds to shifting the level curve defined by (7) to the north-east. Since $k(v)$ remains unchanged, it is easily seen that v_1^c and v_2^c must increase.

On the other hand, an increase in r leads k to move downwards, in the interior (at the end-points, $k(0) = 0$ and $k(\overline{v}_2) = \overline{v}_1$ regardless of r). This follows from the fact that the right hand side of (6) is increasing in r for any $v \in (0, \overline{v}_2)$. Hence, when r increases the left hand side must increase as well, to maintain the equality. This necessitates that k declines.

However, since $k(v)$ decreases when r increases, it must also be the case that v_1^c and v_2^c change. If the increase in r comes from an increase in r_1, the level curve in Figure 1 is unchanged. As seen from Figure 1 or (7), v_2^c increases while v_1^c decreases whenever $a > 0$. Hence, bidder 2 is more likely to stay out, while bidder 1 is less likely to be satisfied with a bid of zero. The reason for the latter is that bidder 1 will exploit his increasingly advantageous position; he is more likely to be active and press his advantage.

In summary, changing a has no effect on the allocation among the types that continue submitting strictly positive bids. Hence, the head start affects the allocation only at "the bottom" (low types). In contrast, rewarding bidder 1 by increasing r_1 affects the allocation everywhere else. Thus, the two ways of favouring a bidder lead to very different outcomes. Below, we summarize the discussion thus far.

Proposition 1 *v_1^c and v_2^c are increasing in a. The former is (weakly) decreasing in r_1, while the latter is (weakly) increasing in r_1. Finally, $k(v)$ is independent of a, but decreasing in r, for all $v \in (0, \overline{v}_2)$.*

The final possibility is that r increases due to a decrease in r_2. This has two partially confounding effects. First, k shifts down. Second, the cost of obtaining a score of a_1 increases for bidder 2, meaning that the level curve in Figure 1 shifts to the right. Thus, v_2^c increases, but v_1^c may increase or decrease. To avoid this complication we will fix the values of a_2 and r_2 by normalizing

$$a_2 = 0, \ r_2 = 1.$$

In the following, when we increase a or r it should be understood that we refer to an increase in a_1 or r_1, respectively. The choice of a is a choice of where to locate the level curve, while the choice of r is a choice only of how much to manipulate k.

2.2 Equilibrium strategies

For the purposes of this paper, the equilibrium strategies themselves are of limited interest. For completeness, the following Proposition describes how a and r impact the equilibrium bids. Since this will not be used in the rest of the paper, the proof is in the Appendix.

Proposition 2 *Increases in a and r change equilibrium bids in the following way:*

1. *If a increases, bidder 1 bids less aggressively. Bidder 2 is more likely to stay out of the auction, but if he does participate he participates with a more aggressive bid.*

2. *If r increases, bidder 1 bids more aggressively if his type is low but less aggressively if his type is high. If bidder 2 continues to participate when r is increased, then he submits a less aggressive bid if his type is low, but a more aggressive bid if his type is high. However, when $a > 0$, bidder 2 participates for fewer types the higher r is.*

Proof. See the Appendix. ■

Since bidder 2 participates less often but bids more when he does participate, the effect of an increase in a on the ex ante expected payment of bidder 2 is *ambiguous*. However, when a is small the latter effect can be shown to dominate and a slight increase in a will *increase* the expected payment of bidder 2. In contrast, the ex ante expected payment from bidder 1 unambiguously *declines*.[12] Thus, if a head start is to increase expected revenue, it must be because it spurs the disadvantaged bidder to bid more aggressively, and this must outweigh the declining payment from the advantaged bidder. When the problem is phrased this way, it appears the seller faces a trade-off. However, in the next section we will show that the seller benefits from introducing a small head start to bidder 1. To explain why this is the case, we rephrase the problem in that section in a way that makes it clear that the apparent trade-off is a function only of the way in which the problem is currently phrased.

We highlight the feature that an increase in r leads bidder 1 to become active for more types. That is, he submits positive bids for *more* types. In other words, he will start bidding *more aggressively* if his type is close to v_1^c. Hence, contrary to the case of a head start, *a bidder will not necessarily lower his bid or effort when he is favoured more*. He may be enticed to increase his bid or effort. In Section 5, this result is contrasted to the prediction of a complete information model.

3 Optimal head starts and handicaps

Assuming the seller's objective is to maximize expected revenue (the expected sum of payments or bids), we now discuss the optimal choice of a and r.[13]

[12]However, in Section 5 it is observed that this result is not robust if more than two bidders participate in the auction.

[13]In this paper the seller can use no other instruments, such as minimum bids or caps. In many of the contests mentioned in the introduction, a winner must be found. In such situations, a minimum bid is not credible. Caps are briefly discussed in Section 5.

We emphasize the significance of the fact that the two instruments change the allocation in different ways. To highlight why this is important, it is useful to follow Myerson (1981) in calculating expected revenue. He defines

$$J_i(v) = v - \frac{1 - F_i(v)}{f_i(v)}, \quad v \in [0, \overline{v}_i],$$

as bidder i's *virtual valuation*. Bulow and Roberts (1989) argue that virtual valuation can be compared to marginal revenue in the standard monopoly problem (see Section 5). Then, Myerson (1981) shows that revenue in any mechanism can be written as the expected value of the virtual valuation of the winner. To calculate this, we need bidder i's winning probability, $q_i(v)$, $i = 1, 2$. In the all-pay auction with a head start and a handicap these are

$$q_1(v|a,r) = \begin{cases} F_2(v_2^c) & \text{if } v \in [0, v_1^c) \\ F_2(k^{-1}(v)) & \text{otherwise} \end{cases},$$

and

$$q_2(v|a,r) = \begin{cases} 0 & \text{if } v \in [0, v_2^c) \\ F_1(k(v)) & \text{otherwise} \end{cases},$$

respectively. Recall that v_1^c and v_2^c depends on a and r. Expected revenue can now be written as

$$ER(a,r) = \int_0^{\overline{v}_1} J_1(v) q_1(v) f_1(v) dv + \int_0^{\overline{v}_2} J_2(v) q_2(v) f_2(v) dv. \tag{8}$$

As a point of comparison to the all-pay auction, consider the revenue maximizing mechanism (among mechanism where the good is sold with probability one). In an optimal mechanism, the seller would maximize the expected value of the virtual valuation of the winner.

Let $\kappa(v)$ denote the correspondence satisfying $J_1(\kappa) = J_2(v)$. κ is a strictly increasing function if J_1 and J_2 are strictly monotonic.[14] Then, κ effectively describes the revenue maximizing mechanism among the set of mechanism where the good is sold with probability one. Specifically, bidder 1 should win the auction if his type exceeds κ when his rival has type v, since it would then be the case that $J_1 > J_2$. Otherwise he would lose. Clearly, such a rule would maximize the expected value of the virtual valuation of the winner.

[14]For expositional simplicity, we will maintain the assumption that virtual valuations are monotonic. However, this assumption is not important.

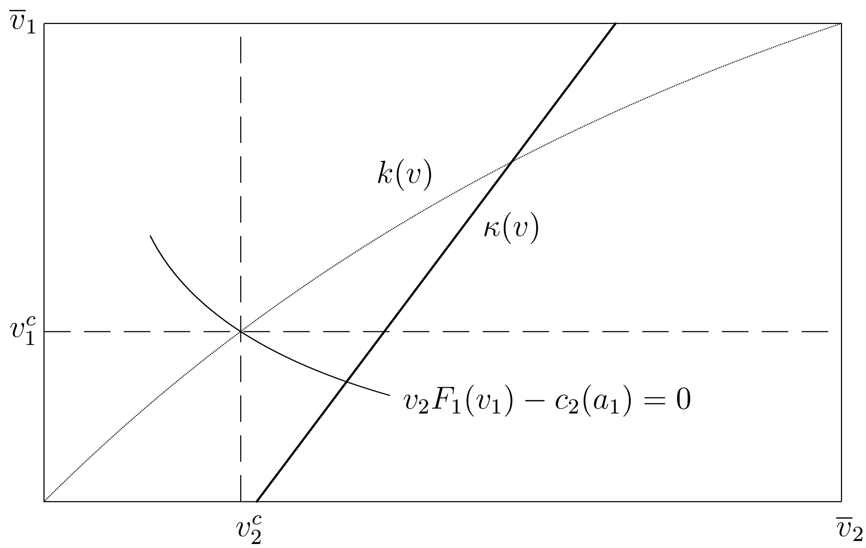

Figure 2: The optimal mechanism ($\kappa(v)$) versus the all-pay auction ($k(v)$).

Figure 2 illustrates; in an optimal mechanism, bidder 1 wins if the combination of types is above $\kappa(v)$, and bidder 2 wins below $\kappa(v)$. Note that the first part of Assumption B implies that $J_1(0) > J_2(0)$, and so it must be the case that $\kappa(v)$ intersects the horizontal axis for some $v > 0$. Moreover, the fact that $J_i(\overline{v}_i) = \overline{v}_i$, $i = 1, 2$, implies that if $\overline{v}_2 > \overline{v}_1$ then $\kappa(v) = \overline{v}_1$ for some $v < \overline{v}_2$, as illustrated in Figure 3. Likewise, if $\overline{v}_2 = \overline{v}_1$ then Assumption B implies that $\kappa'(\overline{v}_2) = 1 < k'(\overline{v}_2)$ when $r = 1$, meaning, once again, that k lies below κ for large v.

Now, it is obviously the case that the standard all-pay auction is far from optimal; $k(v)$ does not coincide with $\kappa(v)$. The problems with the all-pay auction are more pronounced near the bottom-left corner (when both bidders have low types) and the top-right corner (when both bidders have high types). In particular, *bidder 2 wins more often than is optimal near the bottom, while bidder 1 wins more often than is optimal near the top.* Now, changing a and r can be seen as an exercise in manipulating the allocation to bring it as close to the optimal allocation (κ) as possible. As mentioned earlier, an increase in a corresponds to pushing the level curve to the north-east, while an increase in r is equivalent to pushing down $k(v)$.

First, we show that it is generically profitable to give a *head start* to bidder 1. Then, we argue that introducing a *handicap* involves a trade-off. Hence, it is not obvious which bidder should be handicapped. Nevertheless, in the absence of a head start ($a = 0$), Clark and Riis (2000) show that the strong bidder should be handicapped ($r > 1$) in a model where bidders draw valuations from different uniform distributions. However, this result is not robust.

Finally, we observe that the optimal values of a and r are interrelated. Reexamining the model of Clark and Riis (2000), we show that when the asymmetry is sufficiently large, it is the *weak bidder* who should be handicapped, $r < 1$. Thus, the weak bidder would be given a head start, but, at the same time, he would be handicapped.

3.1 Head starts

Compared to the optimal mechanism, one of the drawbacks of the all-pay auction is that bidder 2 wins too often near the bottom-left corner (when types are small). The head start to bidder 1 addresses this problem, as it leads bidder 1 to win when both bidders have low types. Hence, the allocation will move closer to what is optimal. Note that when the problem is phrased in this way – in terms of selecting different mechanisms (all-pay auctions with or without a head start) – there is *no* trade-off associated with introducing a head start.

Theorem 1 *Regardless of r, a small head start to bidder 1 ($a > 0$) will increase expected revenue.*

Proof. Given (8), we observe that

$$\frac{\partial ER(a,r)}{\partial a} = f_2(v_2^c)F_1(v_1^c)\frac{\partial v_2^c}{\partial a}\left(\int_0^{v_1^c} J_1(v)\frac{f_1(v)}{F_1(v_1^c)}dv - J_2(v_2^c)\right), \tag{9}$$

the sign of which is determined by the term in parenthesis (v_2^c is increasing in a). As a approaches 0, v_1^c and v_2^c approaches 0 and this term converges to

$$-\frac{1}{f_1(0)} + \frac{1}{f_2(0)},$$

by L'Hôpital's rule. This is positive, by Assumption B. Hence, $ER(a,r)$ is strictly increasing in a when a is small. ∎

The implication of Theorem 1 is that the seller can benefit from handicapping *some* bidder except possibly in the special case where $f_1(0) = f_2(0)$. This occurs, for example, when bidders are symmetric. However, it is arguably rarely realistic to assume that bidders are symmetric. Thus, we conclude that head starts can be a very useful tool in practice to increase revenue.

3.2 Handicaps

As mentioned earlier (see the discussion surrounding Figure 2), *bidder 1 wins too often when his type is high, but not often enough when his type is low* in the standard all-pay auction compared to the optimal mechanism. Now, consider the effect of increasing r, i.e. handicapping bidder 2 further. $k(v)$ moves down, in the interior. Consequently, in the absence of a head start, bidder 1 wins more often, *regardless* of his type. This moves the allocation *closer* to the optimal allocation if his type is low, but *farther* away if his type is high. Hence, a trade-off exists, and it is not obvious that handicapping the strong bidder is optimal. This is illustrated in the following examples. In Section 4 we perturb the model and establish a whole class of situations where it is the *weak* bidder who should be handicapped.

Example 1: Assume that $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = 1, 2$, with $\overline{v}_2 > \overline{v}_1$ (uniform distributions). Clark and Riis (2000) show that it is profitable to handicap the strong bidder ($r > 1$) when head starts are not allowed.[15] $\square$

Example 2: The strong bidder draws a type from the uniform distribution $F_2(v) = v$, with density $f_2(v) = 1$, $v \in [0, 1]$. In contrast, bidder 1's type is drawn from a distribution function with density

$$f_1(v) = \begin{cases} \frac{1}{2} + 100\left(\frac{1}{10} - v\right) & \text{if } v \in \left[0, \frac{1}{10}\right] \\ \frac{1}{2} & \text{if } v \in \left(\frac{1}{10}, 1\right] \end{cases} .$$

It may be useful to think of f_1 as being obtained by taking f_2 and "shaving off" density on the entire support. This probability mass is then moved to the beginning of the support. In contrast, in Clark and Riis (2000), F_1 is a "scaled down" version of F_2, created by compressing the support. The optimal value of r is approximately 0.61 (when $a = 0$), meaning that it is the strong bidder (bidder 2) that should be favoured.[16] $\square$

Example 2 is an "approximation" of the perturbed model in Section 4 that fits the current model. The intuition behind the result in Example 2 is explained in more detail in that section.

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[15] However, they argue that the reason is that handicapping the strong encourages the weak to bid more. While Clark and Riis (2000) only examine the aggregate revenue, it can in fact be shown that the seller earns more on the strong bidder (in expectation). However, the seller may or may not earn more on the weak bidder (depending on how heterogeneous the bidders are).

[16] The details of the example are omitted, but are available upon request.

3.3 Head starts and handicaps

The advantage of handicapping the strong bidder is that it leads the weak bidder to win more often if his type is low. However, this could also be achieved by giving the weak bidder a head start. The latter option would not suffer the drawback that is associated with a handicap, namely that the weak bidder would win even more often if his type is high.

Once a head start is used to bring the allocation *near the bottom* closer to what is optimal, there is less of an incentive to handicap the strong bidder. Instead, it may be better to use the handicap as an instrument to bring the allocation closer to what is optimal *near the top.* To illustrate this most forcefully, consider once again the model studied by Clark and Riis (2000). Recall that if $a = 0$, the optimal value of r is above one. However, when a and r are chosen jointly to maximize revenue, we see that if the asymmetry is sufficiently large, the weak bidder is simultaneously given a head start ($a > 0$) and a handicap ($r < 1$).

Example 3: Assume that $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = 1, 2$, with $\overline{v}_2 = 3$, $\overline{v}_1 = 1$ (uniform distributions). Then, $ER(a, r)$ is depicted in Figure 3. When $a = 0$, the optimal value of r is approximately 3.6042 ($a = 0$ is captured by the solid curve along the r axis). When $r = 1$, the optimal value of a is approximately 1.0721 ($r = 1$ is captured by the solid curve in the interior). When a and r are chosen simultaneously to maximize $ER(a, r)$, the optimal value of a is above 1.0721, and the optimal value of r is below 1 (and strictly positive). Note that head starts contribute much more to expected revenue than handicaps. $\square$

To appreciate the advantages of handicapping the *weak* bidder, note that in the limit as $r \to 0$, the weak bidder is handicapped so much that he will not submit positive bids. Thus, he will score a. Then, from the strong bidder's point of view, a functions as a *reserve price.* The strong bidder would then win if his type is above a, and otherwise the weak bidder wins. If a is chosen judiciously, then *this mechanism maximizes the payment that is obtainable from the strong bidder.*[17] If he is very strong compared to the weak bidder, it is intuitive that it is worthwhile sacrificing revenue on the weak bidder (who pays nothing in the limiting case) to get more out of the strong bidder. Note that this intuition does not rely on both bidders drawing types from uniform distributions.

[17] This occurs when J_2 is strictly monotonic and $a = v_2^*$, where $J_2(v_2^*) = 0$. In this case, the strong bidder wins if, and only if, his virtual valuation is non-negative.

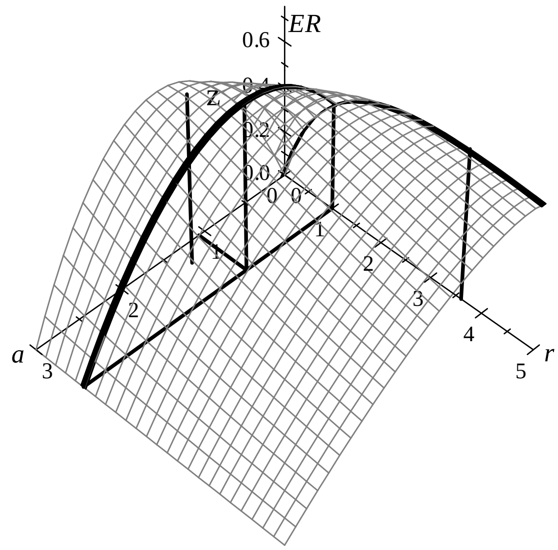

Figure 3: $ER(a, r)$ in the uniform model with $\overline{v}_1 = 1$, $\overline{v}_2 = 3$.

Example 4: Assume that $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = 1, 2$, with $\overline{v}_2 = 5$, $\overline{v}_1 = 1$ (uniform distributions, very large heterogeneity). In this case, when a and r are chosen simultaneously, the optimal value of a is 2.5 and the optimal value of r is zero (corner solution). Expected revenue is 1.25, all of it from the strong bidder, which exceeds what the weak bidder is willing to pay.

4 Perturbation of the standard auction model

We now present a perturbed version of the model in which there is no trade-off associated with the use of a handicap.[18] Specifically, we allow bidder 1 to be *potentially uninterested* in the prize. That is, we assume

$$F_1(v) = 1 - \alpha + \alpha F_2(v),\ v \in [0, \overline{v}_2],$$

for some $\alpha \in (0, 1)$. Thus, bidder 1 is believed to be like bidder 2 with probability α, but to be uninterested in the prize with probability $1 - \alpha > 0$. Obviously, F_2 first order stochastically dominates F_1. It is readily checked that $J_1(v) = J_2(v)$, or $\kappa(v) = v$, when bidder 1 is potentially uninterested. This is the critical feature of the perturbed model.

[18]This section is inspired by Maskin and Riley's (2000) proof that a second price auction may be more profitable than a first price auction.

The derivation of equilibrium in Section 2 remains valid, meaning that

$$\alpha \int_{k(v)}^{\overline{v}_2} \frac{f_2(x)}{x} dx = r \int_{v}^{\overline{v}_2} \frac{f_2(x)}{x} dx.$$

Clearly, $k(v) < v$, for $v \in (0, \overline{v}_2)$, whenever $r > \alpha$. In particular, this is the case when $r = 1$. Thus, in the absence of head starts and handicaps, bidder 1 is more aggressive than bidder 2 for comparable types. However, this outcome is *unequivocally* negative in the current model, since it implies that $k(v) < v = \kappa(v)$, for all $v \in (0, \overline{v}_2)$. *Hence, bidder 1 wins too often compared to what is optimal, regardless of his type.* However, by setting $r = \alpha < 1$, we obtain $k(v) = v = \kappa(v)$. In other words, it is profitable to handicap the weak bidder.

Proposition 3 *Assume that J_2 is strictly increasing and that $a = 0$. The seller profits from handicapping the* ***weak*** *bidder ($r < 1$) if he is potentially uninterested. The optimal value of r is $r^* = \alpha$.*

Proof. In the text. ■

In this model, note that when the weak bidder is handicapped, the economic *environment* may *appear* to be more uneven, but the *outcome* actually becomes more "even", at least in the sense that the bidder with the highest valuation wins. That is, it is efficient to handicap the weak bidder.

Interestingly, even when $r = \alpha$ there is in fact room for a head start in this model. The reason is that when bidder 1 has a mass point in his distribution, $J_1(0) = -\frac{1}{f_2(0)}$ does not accurately reflect the virtual valuation or marginal revenue of bidder 1 when he is uninterested (has type $v = 0$). As we explain in Section 5, in such situations it is profitable to let the bidder win with positive probability even if his type is zero. This can be achieved by giving bidder 1 a head start. Indeed, the all-pay auction with a carefully chosen head start to bidder 1 *and* a handicap of $r^* = \alpha$ maximizes expected revenue among *all* mechanisms in which the prize is sold with probability one. Thus, bidder 1 would once again have a head start and face a handicap.

However, the argument relies on the implicit assumption that bidder 1 actually *participates* when he is uninterested. Since he is uninterested, however, it is not implausible that he will choose to simply stay away from the auction altogether. In this case, a head start to bidder 1 would decrease expected revenue when $r = \alpha$. Hence, the equilibrium selection problem makes recommending a head start to bidder 1 risky. See Section 5 for a related discussion.

5 Discussion

In the following we discuss the intuition in language familiar from the standard monopoly problem. Then, we consider an alternative interpretation of the model, another way of manipulating the contest, and the consequences of allowing more bidders in the auction. Finally, we discuss the incomplete information assumption.

5.1 Monopoly pricing

Bulow and Roberts (1989) argued that the problem facing an auction designer with weak and strong bidders is similar to the problem facing a monopolist with a weak and a strong market and a random capacity. Recall that in the all-pay auction, the weak bidder wins relatively often if types are high, but less often when types are low.

This outcome roughly corresponds to the following policy by a monopolist: In the event capacity is low, sell at a discount on the weak market, but if capacity is large, sell at a discount on the strong market. Given the familiar textbook explanation of third degree price discrimination, this policy is suspect.

First, if capacity is large, it is inoptimal to give a discount to the strong market. It would be better to commit to not selling too much on the strong market, which can be achieved by dumping goods on the weak market. This is essentially what the head start to the weak bidder achieves in the context of the all-pay auction, since it rules out that bidder 2 with type below v_2^c wins.

Second, if capacity is low, it is actually not optimal to give a discount to the weak market, since this consists of consumers with low willingness-to-pay. Rather, the monopolist should sell exclusively on the strong market (remember that marginal revenue on the first unit coincides with the willingness-to-pay of the most eager consumer in the market; $J_2(\overline{v}_2) > J_1(\overline{v}_1)$ if $\overline{v}_2 > \overline{v}_1$). When the auctioneer handicaps the weak bidder, this is the direction in which he moves since it makes it less likely that the weak bidder wins if both have high types.

We next give a more detailed explanation of why a head start to bidder 1 is profitable in the perturbed model of Section 4. Remember that bidder 1 is potentially uninterested, which in the monopoly interpretation corresponds to the weak market having a mass of consumers with zero willingness-to-pay. Figure 4 illustrates revenue, $R(q)$, to a monopolist who faces a market where half of the consumers have positive willingness-to-pay, but the remaining half of the market is unwilling to pay anything. Clearly, once the monopolist starts selling to this group, revenue stays constant, at zero. Marginal revenue is zero, rather than equal to the marginal revenue of the last interested consumer (the tangent line in the Figure). $J_1(0)$ measures the latter.

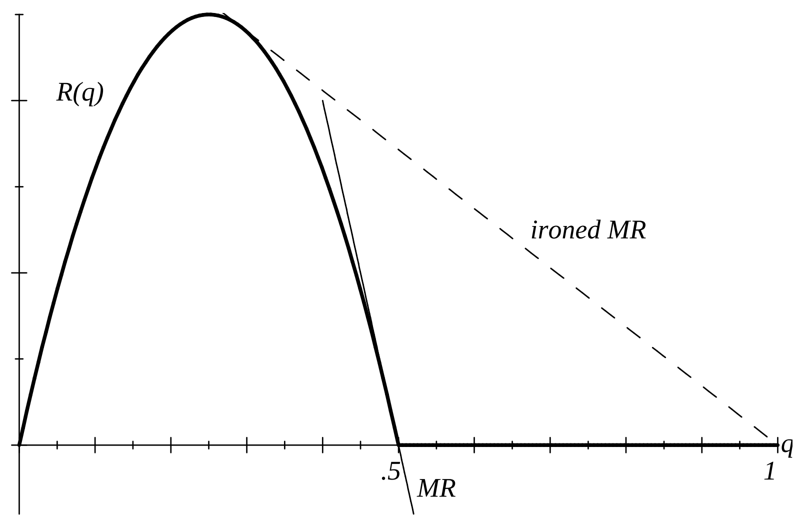

Figure 4: Revenue in a market with a mass of uninterested consumers.

Consequently, marginal revenue is *non-monotonic* in this case. It is positive for small quantities, becomes negative for quantities in the medium ranges, and ends by being zero for large quantities. Then, a monopolist who, for some reason, has to sell a large quantity is better off *ironing* the marginal revenue curve (see Myerson (1981) and Bulow and Roberts (1989)). This can be done by offering two options to consumers: (1) the consumer may pay a high price and get a unit with certainty, or (2) he can sign up for a lottery in which he wins a unit with some probability (less than one) and pays absolutely nothing. Consumers with high valuations, and high marginal revenue, self-select the first option, while everybody else selects the second option. Consumers in the latter group evidently have the same probability of obtaining a unit irrespective of their valuations. The dashed line in Figure 4 represents their "average", or ironed, marginal revenue. Note that when ironing is performed optimally, the average/ironed marginal revenue of those consumers who pick the second option is equal to the marginal revenue of the consumer who is just indifferent between the two options.

Similarly, if the weak bidder benefits from a head start in an all-pay auction, all the types that score a wins with the same probability (i.e. if bidder 2 does not participate). Thus, a head start to bidder 1 irons his marginal revenue curve, and will therefore be profitable when he is potentially uninterested. More formally, the last term in (9) can be written

$$\left(0 \times \frac{F_1(0)}{F_1(v_1^c)} + \int_0^{v_1^c} J_1(v)\frac{f_1(v)}{F_1(v_1^c)}dv\right) - J_2(v_2^c),$$

where the first term is essentially the ironed (expected) marginal revenue of bidder 1 when his type is zero or between zero and v_1^c. Note that as $a \to 0$ ($v_1^c \to 0$), the first term converges to zero (since $F_1(0) > 0$), and the overall expression becomes positive (since $J_2(0) < 0$). Hence, a head start to bidder 1 is profitable.

5.2 Head starts and participation fees

A head start to bidder 1 is isomorphic to bidder 2 having to pay a fixed fee, a, to be allowed to participate or bid.[19] In this interpretation, it is perhaps less surprising that the jointly optimal (a, r) pair may satisfy $a > 0$, $r < 1$. In particular, this would mean bidder 2 has to pay an up-front fee, but in exchange he is rewarded on the margin; additional payments are viewed more favorably.

Moreover, note that demanding a participation fee from bidder 2 eliminates the equilibrium selection problem that arises in the perturbed model when a head start is extended to bidder 1. If bidder 2 does not pay the fee, the prize is simply given to bidder 1 (even if he is uninterested).

5.3 Caps on bids

We have shown that head starts and handicaps affect the auction differently. A head start changes the allocation at the bottom, whereas a handicap's impact is global. There are other ways of manipulating an all-pay auction. For example, the possibility of imposing a cap on bids has been widely studied. This instruments changes the allocation in yet another way, since it is relevant only to bidders with high types.

Che and Gale (1998) consider heterogenous contestants whose valuations are common knowledge (complete information). Gavious et al (2002) assume contestants are ex ante homogenous, but that valuations are private information and that costs are non-linear in the bid. In both models, caps may benefit the recipient of the effort. Sahuguet (2006) complements the two papers by extending the result to the case with ex ante heterogenous contestants, where valuations are private information. However, for tractability, it is assumed that there are exactly two contestants who draw valuations from different uniform distributions, as in the Clark and Riis (2000) model.

Maintaining the assumption of uniform distributions, it can easily be checked that imposing a cap on bids is *not* profitable in the perturbed model of Section 4 when α is high (the bidders are almost symmetric), although it may be profitable

[19] The bidders' scores would be $s_1 = rb_1$ and $s_2 = \max\{b_2 - a, 0\}$, where b_i is bidder i's expenditure, $i = 1, 2$. Bidder 2 wins if $b_2 > a + rb_1$, just as in the original model.

when α is low. Thus, starting from the symmetric uniform model, the profitability of a cap depends on how the asymmetry is modelled.

5.4 Many bidders

When there are two bidders, a bidder who is given a head start responds by lowering his bid. We next present an example designed to show that this prediction is not robust. That is, when there are many heterogeneous bidders it is possible that the bidder who is given a head start will in fact increase his bid. We also argue that head starts may be profitable for more reasons when there are several bidders.

To illustrate this outcome in an extreme case, the following example takes as its starting point the observation by Parreiras and Rubinchik (2006) that when there are many heterogeneous bidders some of them may never participate.

Consider the case where there are 3 bidders. The first bidder is "weak", and characterized by the distribution function $F_1(v)$, $v \in [0, \overline{v}_1]$. The remaining bidders are homogeneous, with types drawn from $F_2(v)$, $v \in [0, \overline{v}_2]$, with $\overline{v}_2 > \overline{v}_1$. For concreteness, assume $F_2(v) = v^2$, $v \in [0, 1]$. The important property is that $F_2(v)/v$ is increasing.

Intuitively, it is possible that the strong bidders compete so hard that it is not worthwhile for the weak bidder to enter the auction. If the weak bidder does not compete, the bidding strategy of the strong bidders (when $a = 0$, $r = 1$) is

$$b_2(v) = \frac{2}{3}v^3.$$

Obviously, this equilibrium hinges on the weak bidder having no incentive to enter. This condition is hardest to satisfy for type $\overline{v}_1$. If the weak bidder has type $\overline{v}_1$ his expected payoff from bidding $b_2(v)$, and thereby winning if both rivals have type below v, would be

$$\overline{v}_1 v^4 - \frac{2}{3}v^3 = v^3\left(\overline{v}_1 v - \frac{2}{3}\right).$$

We assume that $\overline{v}_1 < \frac{2}{3}$, in which case there is no incentive for the weak bidder to enter the auction, since he would earn negative payoff from doing so.

Now, if the weak bidder is given a *small* head start, he still has no incentive to submit positive bids. Nevertheless, expected revenue will increase. The reason is that the weak bidder scores a, meaning that a minimum bid is essentially imposed on the strong bidders. In this model, it is well known that minimum bids are profitable. The reason is that it excludes bidders with negative virtual valuation; $J_2(v) < 0$ when v is small, while bidder 1's expected virtual valuation is zero.

Consider next what would happen if the weak bidder is given a *large* head start. If he continues to bid zero, a still functions as a minimum bid. In this case, the strong bidders' bidding strategy is

$$b_2(v) = \frac{2}{3}v^3 + \frac{1}{3}a,$$

when $v \in \left[a^{\frac{1}{3}}, 1\right]$. To win the auction with probability one, the weak bidder with type $\overline{v}_1$ would have to bid $b_2(1)$, less a. The resulting payoff is

$$\overline{v}_1 - \frac{2}{3}(1-a),$$

while the payoff from bidding zero (scoring a) is $\overline{v}_1 a^{\frac{4}{3}}$, since he would win if both competitors have type below $a^{\frac{1}{3}}$. The former exceeds the latter if a is sufficiently high and $\overline{v}_1 > \frac{1}{2}$. Thus, if $\overline{v}_1 = .6$, for example, it is *not* an equilibrium for the weak bidder to bid zero. Athey (2001) establishes that an equilibrium exists in a large class of games encompassing the current game. Consequently, an equilibrium exists and it must involve positive bids by the weak bidder. In other words, in the current example, the weak bidder *increases* his bid when he is given a sufficiently large head start.

5.5 Favoritism by excluding rivals

Consider the previous example, but assume now that there are *many* weak bidders. In the benchmark auction, where the weak bidders stay out, expected revenue is $\frac{8}{15} \approx 0.533$. The maximal expected revenue that can be obtained from a mechanism where only the two strong bidders are active is approximately 0.585. However, if *all* the weak bidders are given very large head starts, say $a > 1$, the strong bidders will never find it profitable to enter the auction. The auction will therefore be a standard all-pay auction among a large set of weak bidders (their head starts cancel out). As the number of weak bidders grow, however, expected revenue converges to $\overline{v}_1$, which may very well exceed 0.585. In this example, the primary role of the head start is to *exclude* the strong bidders. Baye, Kovenock, and de Vries (1993) were first to prove that this may be profitable in an all-pay auction with complete information. Of course, excluding the strong bidders can be seen as a (somewhat extreme) way of favouring the weak bidders.

5.6 Complete versus incomplete information

A number of important results in the literature on all-pay auctions with heterogenous bidders were first established in the complete information environment, i.e. in a model in which bidders' valuations are fixed and commonly known. The papers by Che and Gale (1998) and Baye, Kovenock and de Vries (1993) mentioned in Section 5.3 and Section 5.5, respectively, are prime examples.

In contrast, this paper focuses on all-pay auctions with incomplete information. This assumption is, in fact, critically important for some of the results. In comparison, consider the consequences of a handicap in a complete information setting where bidder i is known to have valuation v_i, $i = 1, 2$, with $0 < v_1 < v_2$. As long as bidder 2 is not handicapped too much (in particular, as long as $rv_1 < v_2$), it is easily verified that the two bidders use mixed strategies, picking bids according to the distribution functions

$$\begin{aligned} P_1(b) &= \frac{v_2 - rv_1}{v_2} + \frac{r}{v_2} b,\ b \in [0, v_1] \\ P_2(b) &= \frac{1}{rv_1} b,\ b \in [0, rv_1], \end{aligned}$$

for bidder 1 and 2, respectively. Importantly, if r decreases both bid distributions become stochastically weaker, meaning that bidders are more likely to submit low bids. Thus, expected revenue decreases. In particular, *expected revenue unambiguously decreases if the weak bidder is handicapped.* Instead, it is profitable to handicap the strong bidder.

In the model with incomplete information, however, a change in r does not lead to a stochastic deterioration or improvement in the bid distributions. As is evident from the second part of Proposition 2, the new bid distributions must cross the old ones, and it can therefore not be ruled out that it is profitable to handicap the weak bidder. The comparative statics are sensitive to the assumptions about the information structure. The difference in response can be traced back to the fact that the handicap leads to different strategic considerations depending on the bidder's type, and that, contrary to the complete information game, the bidders' types are distributed over an interval in the incomplete information game.

6 Conclusion

We considered contests or all-pay auctions with head starts and handicaps. It was pointed out that they affect the auction in different ways. Thus, one is not a substitute for the other, and it is generally profitable to use both instruments. In the benchmark model, the intuitive results holds that it is optimal to give the weak bidder a head start. However, it is not generally true that the seller profits from handicapping the strong bidder. The use of a handicap entails a trade-off, and we showed it can go both ways. Thus, it is possible that it is the weak bidder who should be handicapped. This is even more likely to be the case when head starts and handicaps can be used simultaneously. In this case, the weak bidder may be given a head start *and* a handicap. We also considered a perturbed model where there is no trade-off associated with using a handicap. In this model it is unambiguously the weak bidder who should be handicapped.

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Appendix: Proof of Proposition 2.

Preliminary step: Scores and Bids. To derive the score obtained by bidder 2 with valuation $v > v_2^c$, using the inverse function theorem on (3), for $j = 2$, yields

$$\frac{ds_2}{dv} = rk(v)f_2(v).$$

Since $s_2(v_2^c|a, r) = a$, it follows that

$$s_2(v|a,r) = a + \int_{v_2^c}^{v} rk(x)f_2(x)dx, \tag{10}$$

for $v \in [v_2^c, \overline{v}_2]$. Bidder 2's bid, $b_2(v|a,r)$, equals his score (recall the normalization $a_2 = 0$, $r_2 = 1$). If bidder 2's type is below v_2^c he stays out of the auction or scores zero.

As mentioned, in equilibrium bidder 1 with type v scores the same as bidder 2 with valuation $k^{-1}(v)$. Alternatively, we can derive $s_1(v)$ in the same manner $s_2(v)$ was derived,

$$s_1(v|a,r) = a + \int_{v_1^c}^{v} k^{-1}(x)f_1(x)dx, \tag{11}$$

for $v \in [v_1^c, \overline{v}_1]$. Given (1), bidder 1's bid is

$$b_1(v|a,r) = \begin{cases} 0 & \text{if } v \in [0, v_1^c] \\ \int_{v_1^c}^{v} \frac{1}{r}k^{-1}(x)f_1(x)dx & \text{otherwise} \end{cases}. \tag{12}$$

Proof of part 1: When a increases, v_2^c increases as well, meaning that bidder 2 stays out for more types. To see how his bid changes for the active types, note that

$$\frac{\partial b_2(v|a,r)}{\partial a} = \frac{\partial s_2(v|a,r)}{\partial a} = 1 - rv_1^c f_2(v_2^c)\frac{dv_2^c}{da}.$$

Moreover,

$$\begin{aligned} \frac{dv_2^c}{da} &= \frac{1}{F_1(v_1^c) + v_2^c f_1(v_1^c)k'(v_2^c)} \\ &= \frac{1}{F_1(v_1^c) + rv_1^c f_2(v_2^c)}, \end{aligned} \tag{13}$$

where the first equality follows from (7) and the second from (5). Hence,

$$\frac{\partial b_2(v|a,r)}{\partial a} = \frac{F_1(v_1^c)}{F_1(v_1^c) + rv_1^c f_2(v_2^c)} \geq 0 \tag{14}$$

for $v \in (v_2^c, \overline{v}_2]$. Thus, the active types responds by bidding *more aggressively*. Although the bid (or score) increases, it increases by less than a increases, as the derivative is strictly less than 1. Bidder 1 bids less aggressively, which follows directly from (12) when we recall that v_1^c increases and r and k^{-1} are unchanged.

Proof of part 2: If $a = 0$, note that implicit differentiation of (6) reveals that

$$\frac{dk}{dr} = -\frac{k}{f_1(k)} \int_v^{\overline{v}_2} \frac{f_2(x)}{x} dx = -\frac{k}{f_1(k)} \frac{1}{r} \int_k^{\overline{v}_1} \frac{f_1(x)}{x} dx,$$

where (6) was used to obtain the last equality. It follows that

$$\frac{drk}{dr} = k + r\frac{dk}{dr} = k \left[1 - \frac{1}{f_1(k)} \int_k^{\overline{v}_1} \frac{f_1(x)}{x} dx \right].$$

This is negative if k is small since the term in the bracket goes to $-\infty$ as $k \to 0$. Consequently, bidder 2's bid, $s_2(v|0,r)$, decreases in r for small types. The fact that bidder 1 bids more aggressively for small types can be proven in a similar manner. If $a > 0$, the result follows from (10) and (12) coupled with the fact that v_1^c is decreasing in r, while v_2^c is increasing in r.

Turning to high types, the highest score submitted by bidder 2 is

$$\overline{s}_2 = s_2(\overline{v}_2|a,r) = a + \int_{v_2^c}^{\overline{v}_2} rk(x) f_2(x) dx.$$

Since the bidders share the same maximal score, $\overline{s}_1 = \overline{s}_2$, we infer that bidder 1's maximal bid is

$$\overline{b}_1 = c_1(\overline{s}_2) = \frac{\overline{s}_2 - a}{r} = \int_{v_2^c}^{\overline{v}_2} k(x) f_2(x) dx.$$

This is decreasing in r since v_2^c is increasing in r and k is decreasing in r. Hence, bidder 1 bids less aggressively if his type is close to $\overline{v}_1$. In a similar manner we can calculate bidder 2's maximal bid, $\overline{b}_2$,

$$\overline{b}_2 = \overline{s}_1 = s_1(\overline{v}_1|a,r) = a + \int_{v_1^c}^{\overline{v}_1} k^{-1}(x) f_1(x) dx.$$

This increases in r since v_1^c decreases and k^{-1} increases. Thus, bidder 2 bids more aggressively if his type is high. ■