A Note on the Accuracy of Extended-Path Solution Methods for Dynamic General Equilibrium Economies

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Abstract

We show that the deterministic Extended-Path (EP) method of Fair and Taylor (1983) solves standard dynamic stochastic general equilibrium models with similar accuracy to the best results reported in the literature for alternative methods. The EP method demands more computer time than other methods but has offsetting benefits in terms of simplicity and generality that make it an attractive choice.

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1 Introduction

Relatively large Euler-equation errors are reported by Aruoba et al. (2006) for methods of solving dynamic stochastic general equilibrium models which employ linear and log-linear approximations. The inaccuracies introduced by linearization can be severe enough to call the use of these simple and fast methods into question. Non-linear approachs such as projection methods, exhibit much better accuracy but can be complex to implement requiring model-specific decisions over approximation, integration, projection, and equation-solving procedures (Judd, 1998, pp. 606). This paper presents accuracy results for an alternative non-linear procedure, the "extended-path" (EP) method (Fair and Taylor, 1983), which limits these problem-specific issues facing the researcher to equation solving.

Early results on the accuracy of the EP method (Gagnon (1990), and Taylor and Uhlig (1989)) were promising but limited to qualitative comparisons with solutions available from other methods. Here we report Euler-equation errors as computed in Judd (1992). In the benchmark case EP ranks as good or better than the most accurate methods reported by Aruoba et al. (2006). EP also generates extremely uniform accuracy across the state space and for large intervals around the steady-state. The relative performance of the deterministic EP approach does deteriorate for models with extreme non-linearities and extremely volatile stochastic processes, but it remains superior to linearization methods and generalizes easily to a broad set of applications.

Section 2 below outlines the implementation of EP in this paper and briefly discusses some details. Section 3 presents our accuracy results. Section 4 concludes.

2 The Extended-Path Method.

As in Aruoba et al. (2006) we study the basic stochastic neoclassical growth model with leisure which is well known from the macroeconomics literature. There is a single representative household which maximizes expected utility according to;

$$U = E\{\sum_{t=1}^{\infty} \beta^t u(C_t, (1 - l_t))\}. \tag{1}$$

C_t is period-t consumption, and l_t is labour supplied from a unit endowment of time per period. $\beta < 1$ is the household's subjective rate of time preference. The period utility function is isoelastic over consumption and leisure and is parameterized as:

$$\begin{aligned} u(C_t, l_t) &= \frac{1}{1-\tau}(C_t^{\varepsilon}(1 - l_t)^{(1-\varepsilon)})^{(1-\tau)} \quad \varepsilon \in (0, 1), \quad 0 \leq \tau \neq 1 \\ u(C_t, l_t) &= ln(C_t^{\varepsilon}(1 - l_t)^{(1-\varepsilon)}), \qquad \tau = 1. \end{aligned} \tag{2}$$

Output, Y_t, is produced according to,

$$Y_t = A_t S_t K_t^{\alpha} l_t^{1-\alpha}, \tag{3}$$

where $\alpha \in (0, 1)$. $A_t = (1 + g)A_{t-1}$. $g \geqslant 0$ allows for exogenous growth, and $S_t > 0$ is a stochastic productivity process given by,

$$S_t = exp\{z_t - \sigma_{\epsilon}^2/2(1 - \rho^2)\}, \quad z_t = \rho z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.\, N(0, \sigma_{\epsilon}^2)\, \forall t. \tag{4}$$

These assumptions imply that $E[S_t] = 1$. Further the T-step forecast of this

process conditional on z_{t-1} is given by,

$$E[log(S_{t+T}) \mid z_{t-1}\} = \rho^{(T+1)} z_{t-1} - \frac{\sigma_\epsilon^2}{2(1-\rho^2)}, \tag{5}$$

with conditional variance of,

$$Var[log(S_{t+T}) \mid z_{t-1}] = \sigma_\epsilon^2 \sum_{i=0}^{T} \rho^{2i}. \tag{6}$$

Lastly, the usual law of motion for capital implies the following aggregate resource constraint for the economy,

$$Y_t = K_{t+1} - (1-\delta)K_t + C_t. \tag{7}$$

A solution to this problem is a set of decision rules which, given current state-variable values (i.e. the capital stock K_t, and technology level $A_t S_t$), yield the current choice-variable values (i.e. labour supply l_t, and consumption C_t). Such a solution is fully characterized by the aggregate resource constraint (equation 7), the stochastic productivity process (equation 4), and the following two first-order conditions,

$$\frac{(C_t^\varepsilon (1-l_t)^{(1-\varepsilon)})^{1-\tau}}{C_t} = \beta E_t \left[\frac{(C_{t+1}^\varepsilon (1-l_{t+1})^{(1-\varepsilon)})^{1-\tau}}{C_{t+1}} (1 + \alpha A_{t+1} S_{t+1} K_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta) \right] \tag{8}$$

$$\frac{(1-\theta)}{\theta} \frac{C_t}{1-l_t} = (1-\alpha) A_t S_t K_t^\alpha l_t^{-\alpha} \tag{9}$$

Equation 8 is the standard Euler equation relating current and future marginal utilities. Equation 9 is a static first-order condition relating the marginal utili-

ties of consumption and leisure in each period.

As is well known, closed-form analytical solutions for the decision rules of this model do not exist. Rather than solving directly for approximate closed-form decision rules (eg. as with projection methods), the EP method generates time-series that satisfy the model equations and thus characterizes these decision rules indirectly.

Consider first a deterministic case of the model. We have then $N = 4$ coupled first-order difference equations describing time-paths of the model's two choice and two state-variables, $\{l_t, C_t, K_t, A_t S_t\}_{t=0}^{t=\infty}$. Standard solution methods for such systems are readily available[2]. Central to these methods is the choice of how to effectively "close" the open system of difference equations. We employ a "two-point boundary" problem approach that closes the system by imposing the initial conditions given by the model's state variables *and* imposing suitable "terminal" conditions on the model's choice variables at some distant horizon T. This results in a closed system of $N \times (T+1)$ equations that can be efficiently solved to essentially any desired degree of accuracy. Long-run steady-states or balanced-growth variable values are natural candidates for terminal conditions. Given a sufficiently large T (eg. such that $\beta^T \simeq 0$), this method eliminates two layers of iteration (type II and III) relative to the "shooting" type procedure outlined by Fair and Taylor (1983).

Of course, in the stochastic environment we have the added difficulty of computing the conditional expectation on the right side of equation 8. The deterministic EP method simply takes the conditional expectation inside the non-linear functions of this equation and solves at the expected value of the vector of future shocks conditional on current realizations. For example, conditional on the current technology, $A_0 S_0$, equations 5 and 6 can be used to calculate the expected values of future technology levels, $E_{t=0}\{A_t S_t\}_{t=1}^{t=T}$. Substituting these

[2]See Press et al. 1992, Chpt. 17 for a comprehensive discussion and related algorithms.

into the right side of equation 8 yields a deterministic system which is readily solved employing the methods described above. The resulting time-series contains solutions for the current periods decision variables (C_0 *and* l_0) conditional on the expectations assumed and on the current state. This solution also implies the next periods initial condition, K_1, which together with another draw from the stochastic process provides the starting point for finding the next period's decision variables. Iterating on this process q times yields a q-length time-series and the whole procedure can be replicated for an arbitrary number of vectors of realizations.

Since the EP method works straight from the model's first-order conditions and constraints it is directly applicable to systems with externalities, distortionary taxes, and even inequalities. Furthermore, this method does not suffer a "curse of dimensionality" problem making it useful for solving models with multiple state variables.

Both linearization methods and this deterministic EP method rely on the imposition of some version of certainty equivalence to evaluate conditional expectations. In either case the magnitude of the approximation error depends on the degree of nonlinearity of the model. The advantage of the EP method, in this regard, is that the original model structure is retained so that approximation errors arise due to the inaccurate evaluation of conditional expectations only, rather than being further compounded by the model modifications inherent in linearization.

In the next section we demonstrate the high degree of accuracy obtainable with this EP method.

3 Solution Accuracy.

3.1 Euler Equation Errors.

For convenience we express the Euler equation 8 as,

$$0 = \Gamma(C_t, l_t) - \int_{-\infty}^{\infty} \Omega(C_{t+1}, l_{t+1}, K_{t+1}, \epsilon_{t+1}) f(\epsilon_{t+1}) d\epsilon_{t+1}, \tag{10}$$

where $f(\epsilon_{t+1})$ denotes the distribution function for the period-t+1 technology shocks and $\Gamma(\bullet)$, and $\Omega(\bullet)$ just simplify notation. This equation should hold exactly in the model given the current-period state variables, K_t and ϵ_t. Our solutions are only approximations, however, derived by substituting for the integral in this equation with $\Omega(C_{t+1}, l_{t+1}, K_{t+1}, 0)$. Deviations from zero will therefore be observed when evaluated at our approximate solutions. As discussed in Judd (1992) and Aruoba et al. (2006) such deviations are a good measure of solution accuracy.

Since an exact solution for the integral in equation 10 is unknown, we employ the method of Judd (1992) of approximating this integral with an 8-point, fifteenth-order accurate, Gauss-Hermite quadrature rule as follows.

$\epsilon_{t+1} \sim N(0, \sigma_\epsilon^2)$, so employing the variable transformation, $x_{t+1} \equiv \epsilon_{t+1}/\sigma_\epsilon\sqrt{2} \sim N(0,1)$, the integral in equation 10 can be expressed as,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Omega(C_{t+1}, l_{t+1}, K_{t+1}, x_{t+1}) exp(-x_{t+1}^2) dx_{t+1}.$$

This integral is approximated by the following finite summation,

$$\frac{1}{\sqrt{\pi}} \sum_{i=1}^{8} \Omega(C_{t+1}, l_{t+1}, K_{t+1}, x_i) w_i, \tag{11}$$

where x_i, and w_i are Guass-Hermite abscissas and weights.

With this integral approximation in hand the procedure for deriving Euler-

equation errors is straightforward. Given the current-period state variables, K_t and ϵ_t the EP method is employed to obtain solutions for the current-period decision variables, $\{\hat{C}_t, \hat{l}_t, \hat{K}_{t+1}\}$. $\hat{K}_{t+1}$ is then coupled with each of the $\{x_i\}_{i=1}^{i=8}$ abscissa values in succession to form 8 possible t+1-period states conditional on the current-period decision-variable solutions. The model is then solved using the EP method to obtain 8 sets of t+1-period decision-variable solutions, $\{\hat{C}_{i,t+1}, \hat{l}_{i,t+1}, \hat{K}_{i,t+2}\}_{i=1}^{i=8}$. The Euler-equation error associated with the current-period decision-variable solution is then simply calculated as,

$$Eerror_t = \Gamma(\hat{C}_t, \hat{l}_t) - \frac{1}{\sqrt{\pi}} \sum_{i=1}^{8} \Omega(\hat{C}_{i,t+1}, \hat{l}_{i,t+1}, \hat{K}_{t+1}, x_i) w_i.$$

Lastly, following Judd (1992), these optimization errors are expressed as percentages of current-period consumption.

$$Cerror_t = \left[(\Gamma(\hat{C}_t, \hat{l}_t) - Eerror_t)(1 - \hat{l}_t)^{(\varepsilon-1)(1-\tau)} \right]^{\frac{1}{\varepsilon(1-\tau)-1}} \tag{12}$$

3.2 Numerical Results.

For comparability with the results of Aruoba et al. (2006) we employ their benchmark calibration of this model and we also examine the alternative calibrations used by these authors to guage the effects of highly non-linear specifications (high variance/high risk aversion) on solution accuracy. The parameter values for these calibrations are reported in Tables 1 and 2.

Table 1: Benchmark Parameters

Parameter	β	τ	ε	α	δ	ρ	σ_ϵ	g
Value	0.9896	2.0	0.357	0.4	0.0196	0.95	0.007	0.0

Table 2: Alternative Calibrations

Case	$\sigma_\epsilon = 0.007$	$\sigma_\epsilon = 0.035$
$\tau = 2$	Benchmark	Intermediate 3
$\tau = 10$	Intermediate 1	Intermediate 4
$\tau = 50$	Intermediate 2	Extreme

Table 3 reports in base 10 logs the maximum, the average, and sample confidence intervals around the average, of solution errors in consumption equivalent terms from 10,000 simulated observations[3] under each calibration. For comparison purposes the table also reports results from Aruoba et al. (2006).

Table 3: Euler Errors in %Consumption Terms (Abs(log_{10}))

Extended Path	Max.	Avg.	Avg.$\pm 2 std.error$
Benchmark	-5.2505	-5.2704	(-5.2806,-5.2604)
Intermediate 1	-4.8891	-4.9120	(-4.9174,-4.9066)
Intermediate 2	-4.3394	-4.4206	(-4.4388,-4.4032)
Intermediate 3	-3.7793	-3.8730	(-3.8992,-3.8482)
Intermediate 4	-3.3953	-3.5158	(-3.5439,-3.4893)
Extreme	-2.7707	-3.0115	(-3.0566,-2.9707)

Value Function*	Max.	Avg.
Benchmark	-4.4343	-5.6498
Extreme	-4.015	-4.4949

Chebyshev*	Max.	Avg.
Benchmark	-3.3281	-5.4330
Extreme	-2.5269	-4.6578

Log-Linear*	Max.	Avg.
Benchmark	-2.2002	-4.2002
Extreme	-1.4315	-2.6131

* Value-Function iteration, Spectral method with Chebyshev polynomials, and Log-Linear approximation results as reported in Aruoba et al. (2006).

[3]Resulting from 200 replications of 50 quarterly observations. Each calibration experiment employed the same vector of realized shocks.

In general the errors resulting from the EP method are small. The maximum Euler error for the benchmark case implies a \$1 optimization mistake for each \$178,033. As expected this performance deteriorates markedly as the non-linearities and stochastic volatility in the model increase. For the "extreme" calibration, mistakes are on the order of \$1 per \$590. This, however, is no worse than the maximum Euler error reported by Aruoba et al. (2006) for the spectral projection method (Chebyshev) and is significantly better than the log-linear approximation.

There is also relatively little variation in the accuracy of the EP method across the 10,000 simulated observations as indicated by the reported confidence intervals. This is because the method does not rely on discretization or other state-space restrictions nor on approximations relative to a fixed point (steady-state). Thus, while projection methods for example are more accurate on average, particularly under the extreme calibration, the EP method provides uniformly acceptable accuracy across the state-space and far from the steady-state. In anycase the EP method continues to outperform the alternative linearization methods reported in Aruoba et al. (2006).

3.3 Robustness of Results.

We also explored the accuracy of the EP method under various alternative model specifications. A brief summary of the main results is presented in Table 4 below.

The first three rows of Table 4 repeat alternative calibrations studied in Auroba et al. (2006) and confirm their results. The non-linear EP method delivers Euler equation errors near machine zero in the deterministic case. Log-utility and a high return on capital (net MPK = %8.4) yield results very close to those for the benchmark model.

Table 4: Robustness experiments

Model Variation	Max	Avg	Avg.$\pm 2 std.error$
deterministic ($\sigma_\varepsilon^2 = 0$)	-15.017	-15.365	(-15.625,-15.204)
log utility ($\tau = 1$)	-5.299	-5.318	(-5.323,-5.313)
high MPK ($\beta = 0.98$)	-5.112	-5.130	(-5.135.-5.126)
linear ($\tau = 1, \delta = 1$)	-11.484	-11.488	(-11.489,-11.487)
random walk (eqtn:13)	-4.493	-4.504	(-4.506,-4.502)
growth ($g = 0.02$)	-5.173	-5.191	(-5.196,-5.187)

The fourth row of Table 4 shows accuracy close to machine zero for a near-linear parameterization where certainty equivalence applies. The last two rows increase model non-linearities in directions not explored via the intermediate cases above. The fifth row presents results when the exogenous productivity factor, S_t, is assumed to follow a random walk (*i.e.* $\rho \to 1$);

$$ln(S_t) = ln(S_{t-1}) + \xi_{St}, \quad \xi_{St} \sim i.i.d.\ N(0, \sigma_\xi^2). \tag{13}$$

While modestly less accurate, the results for this case remain close to those for the benchmark. The sixth row shows the case allowing for exogenous growth of 4% per annum (setting $g = 0.02$ in the technology process $A_t = (1 + g)A_{t-1}$). Again, the EP method continues to deliver excellent results in line with those found for the benchmark.

4 Conclusion.

This paper shows that the deterministic extended-path method of Fair and Taylor (1983) solves dynamic stochastic general equilibrium models with similar accuracy to the best results reported in the literature for alternative solution methods.

The main disadvantage of the method is its relatively high computing time.

For the benchmark two state-variable growth-model studied here, it takes just over 5 minutes to generate an artificial sample of 10000 observations on a 2.8 Mhz Pentium IV running Linux (kernal 2.63). Thus, while not prohibitive, the use of this solution method for estimation purposes may be computationally extensive.

The relative performance of the method does deteriorate for models with extreme non-linearities and extremely volatile stochastic processes. However, its accuracy remains superior to that of linearization methods, and it continues to yield extremely uniform results across the state space and for large intervals around the steady-state.

Programming and debugging is largely limited to the implementation of standard non-linear solution techniques applied directly to the model's first-order conditions in stationary form. Since the method works straight from first-order conditions it is directly applicable to models with externalities, distortionary taxes, and inequality contraints. Lastly, the method does not suffer a "curse of dimensionality" problem making it useful for systems with multiple state variables.

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