Comparative Statics and Welfare in Heterogeneous Contests: Bribes, Caps, and Performance Thresholds

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Abstract

Comparative statics for contests with two privately informed and ex ante heterogeneous contestants are analyzed. Strategies and payoffs are examined and it is shown that total effort may *increase* when one contestant becomes weaker. The second part of the paper considers dynamic contests in which one bidder may endogenously be revealed to be weak. For example, the first contestant has the possibility of preempting the contest by paying a bribe or taking some other action. If the bribe is not paid the second contestant infers the first contestant is relatively weak or uninterested. Adding the possibility of paying a preemptive bribe decreases expected payoff for a set of types of at least one of the contestants, possibly the contestant who is ostensibly advantaged by the option to preempt the contest. However, the payoff of the recipient of the effort and the *ex ante* payoff of *both* contestants may improve.

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1 Introduction

When the less-likely job candidate is called in for an interview two months after the application deadline has passed he may be justified in thinking that none of the more-likely job candidates impressed the prospective employer. This, in turn, may very well influence how much effort he decides to put into preparing for the interview. Likewise, when the would-be world champion hears rumors that his strongest rival has broken the world record in a training session he may be well advised to reexamine his own commitment. Similarly, the lobbyist might give up hope of winning support for his cause if he learns of the lavish spending of another lobbyist.

These examples illustrate two obvious but nevertheless important points. First, a contestant's perception of the competition is likely to influence his behavior in the contest. Second, a contestant's perception of the competition may change over time. Hence, diffusion of information may have a significant impact on contests that unfolds over time.

Reflecting these two observations we first engage in a comparative statics exercise, examining how the outcome of the contest is affected if one contestant becomes "weaker" in the eyes of his rivals. We then use this analysis as a stepping stone to consider a larger dynamic game in which beliefs may change endogenously over time.

As the contest is modelled as an all-pay auction the first part of the paper is thus related to the small literature on comparative statics in auctions. Lebrun (1998) considers first price auctions where bidders are ex ante *heterogenous* and examines what happens when the beliefs concerning *one* bidder changes. Hopkins and Kornienko (2007) look at first price auctions as well as all pay auction. They assume all bidders are ex ante *homogenous* and that the beliefs about *all* bidders change in the same way, implying bidders are ex ante homogeneous both before and after the change.

As in Lebrun (1998), we assume bidders are ex ante heterogenous and that the beliefs about one bidder changes. However, not only is the auction format different compared to Lebrun (1998), the way in which beliefs change is also different. Lebrun (1998) assumes the old and new valuation distributions (capturing other bidders' beliefs about the bidder) share the same support and that one dominates the other in terms of the reverse hazard rate (implying first order stochastic dominance). In contrast, in this paper we assume the change in beliefs is in the manner of a *truncation*, meaning that the change in beliefs is equivalent to ruling out that the bidder's valuation is above a certain threshold. As mentioned, in the second part of the paper we examine a dynamic game in which a change in beliefs of this nature occurs endogenously.

Due to the difficulty of analyzing heterogenous contests or all-pay auctions we follow Amann and Leininger (1996) in assuming there are exactly two contestants. Most other papers allowing for heterogeneity among contestants, such as Clark and Riis (2000) and Sahuguet (2006), make the same assumption, as does Lebrun (1998).[1] The exception is Parreiras and Rubinchik (2006), who analyze aspects of contests with many contestants.

To summarize a few key results, we show the change in beliefs will result in one of two changes in the outcome of the contest. The contestant whose rival becomes weaker either (i) becomes less aggressive regardless of his type or valuation or *(ii)* he becomes more aggressive if his type is low and less aggressive if his type is high. The opposite holds for the contestant whose distribution was truncated. The former contestant is, perhaps not surprisingly, better off when his competition weakens. Interestingly, the latter contestant may be better or worse off, depending on how his rival reacts to the change in beliefs.

In some cases there may be a third party who benefits from the effort (or bids) of the contestants, and it is therefore also interesting to consider how expected total effort changes. While the expected effort from the contestant who becomes weaker diminishes, the other contestant may or may not increase his effort, in expectation. Hence, it is possible that total effort *increases* when one contestant becomes weaker. This can occur when the two contestants are very heterogeneous at first and the change in beliefs reduces the asymmetry.

In the second part of the paper we argue that information leakage in contests with a *performance threshold* of some sort may give rise to the type of revision of beliefs discussed previously. For instance, a firm might promote an existing employee to a more senior position without any competition if the employee's performance has been sufficiently impressive in the past. If the internal candidate's performance is seen to be lacking, however, the firm might initiate a search for an external candidate. Notice that once the job is posted externally it will be possible to infer that the internal employee did

[1] However, Lebrun (1998) generalizes his results to the case where there are many bidders but each bidder's type is drawn from one of two distributions.

not impress.[2]

A similar situation occurs when one contestant is in a position to offer a bribe to preempt the contest from taking place. If the contest does indeed take place the other contestants will realize that the first contestant did not pay the bribe.

In a situation where a government contract is awarded by a corrupt official, Lien (1990) notes that "in the real world, the corrupt official determines the winner not only by the amount of the bribe, but by some other considerations such as friendship. For example, if the owner of firm i is a close friend, the official will then take this factor into account". It is conceivable that the official will show his friendship by making it known to his friend that an early and relatively low bribe will suffice, yet if such a bribe is not provided, other firms will be given the chance to compete for the contract.[3]

Lastly, contests with caps share the same characteristics when one contestant, before making his own decision, learns whether another contestant's effort or expenditure has or has not met the cap. Che and Gale (1998) and Gavious et al (2002) point out that caps exist in many contests.[4] Examples include salary caps in sports and caps on campaign contributions in elections. However, it is not unheard of for information regarding a team's spending or a politician's campaign financing to leak before the trade deadline or the election date, respectively. Nevertheless, in the existing literature on caps in contests it is assumed that decisions are taken simultaneously, or that there is no information leakage.

In a *simultaneous* contest, Che and Gale (1998) consider *heterogenous* contestants whose valuations (how much they value winning) are *common knowledge*. They show that a cap may in fact increase the sum of the contestants' efforts.[5] Since their focus is on socially wasteful campaign spending, they conclude that caps may be damaging. However, in many contests effort is not wasteful, and there is some party who benefits from the effort. From

[2] See Chan (1996) and Chen (2005) for papers in labor economics dealing with internal and external candidates, but with the assumption of complete information.

[3] See Burguet and Perry (2007) and the references therein for models of bribery in first price auctions.

[4] Gavious et al (2002) also argue that imposed technological constraints in sports (e.g. in Formula 1) is a different form of a cap. We might add that in certain sports the existence of a "perfect score" creates a similar situation.

[5] In Kaplan and Wettstein (2006) and Che and Gale (2006) the assumption that the cap is rigid is relaxed. That is, a contestant can break the cap, but at some additional cost (e.g. a fine).

this individual's point of view, a cap may therefore be beneficial.

Taking this view, Gavious et al (2002) assume contestants are ex ante *homogenous*, but that valuations are *private information*. They also find that caps may benefit the recipient of the effort. Sahuguet (2006) complements the two papers by extending the result to the case with ex ante *heterogenous* contestants, where valuations are *private information*. However, for tractability, it is assumed that there are exactly *two* contestants who draw valuations from different *uniform* distributions.[6]

These assumptions are also used in Clark and Riis (2000), who consider a contest in which contestants are treated differently. Specifically, one of the contestants must "outscore" the other contestant by a certain percentage in order to win.[7] Clark and Riis (2000) show that this type of discrimination may be beneficial to the recipient of the effort. However, if the recipient controls who discrimination is in favour of, he will elect to discriminate in favour of the wrong contestant, from society's point of view.

The current paper shares features of both sets of papers. Specifically, there is a cap or performance threshold in the contest and contestants are treated differently, in this case in the sense that contestants move in *sequence* – in the formal model, the second contestant learns whether or not the first contestant met the cap before making his own decision. The two contestants are allowed to be heterogeneous and valuations are private information.

When the performance threshold is introduced at least one contestant has a set of types for which expected payoff decreases. For example, the first contestant may be made *worse off* if he is given the chance to preempt the contest with a bribe. The reason is that if his type is low he would be unwilling to pay the bribe, yet failure to do so signals that he is weak and the rival may respond by becoming more aggressive.

Even though a subset of types are worse off it is in principle possible that the contestants are better off ex ante. By imposing the assumption in Clark and Riis (2000) and Sahuguet (2006) that the contestants draw types from different uniform distributions we are able to verify that this may indeed occur. Moreover, for this class of distributions there is a cap or bribe such that the total expected effort increases, thus making the recipient of the effort

[6] These two assumptions in conjunction allow contestants' behavior to be described in a closed form. See Amann and Leininger (1996) for a more general treatment of contests with no caps and two heterogenous contestants.

[7] Lien (1990) examined this type of discrimination as well, but he assumed the contestants are homogeneous.

better off. Hence, it is possible that *both* contestants as well as the recipient of the effort are all better off ex ante. Even when this is not the case, the effort-maximizing bribe *improves* social surplus, which is in contrast to the result in Clark and Riis (2000).

The remainder of the paper is organized as follows. The model is presented in Section 2. In Section 3 the comparative statics of the contest are studied. A dynamic game in which beliefs are updated endogenously over time is then presented and analyzed in Section 4. Section 5 concludes.

2 Model

The contest is modelled as an all-pay auction. There are two risk neutral contestants, or bidders. Bidder i's valuation or type is drawn from the continuously differentiable distribution F_i on $v \in [\underline{v}, \overline{v}_i]$, with $\overline{v}_i > \underline{v} \geq 0$, $i = 1, 2$. The density, f_i, is assumed to be finite and strictly positive on $(\underline{v}, \overline{v}_i]$. Given the bidder's type is v, his problem is to decide whether to participate in the auction, and, if so, decide which bid to submit in order to maximize his expected payoff.

Amann and Leininger (1996) showed that bids are non-decreasing in type. More precisely, if a bidder bids above zero in equilibrium, an increase in his type would induce him to bid strictly higher. However, it is possible that exactly one bidder has an interval of types (starting at $\underline{v}$) for which he would not participate. Intuitively, this could occur if one bidder is so intimidated by his competitor that he does not believe it worthwhile to "sink" a bid into the auction. Define v_i^0 as being the infimum of the set of bidder i's types who are active, i.e. bids at least zero, $i = 1, 2$.

More generally, let $\varphi_i(b)$ denote the inverse bid function of bidder i, with $\varphi_i(0) = v_i^0$, $i = 1, 2$.[8] A bid of b will be sufficient to outbid bidder i if bidder i's valuation is below $\varphi_i(b)$, which occurs with probability $F_i(\varphi_i(b))$. Hence, from the point of view of bidder j, $F_i(\varphi_i(b))$ is the distribution of bids that he faces from his competitor.

Another feature of the equilibrium is that both bidders share a common maximal bid, $\overline{b}$. Otherwise, the bidder with the highest possible bid should

[8] If $v < v_i^0$, bidder i's strategy is to "stay out". Technically, it is important no bidder bids zero for a mass of types if $\underline{v} > 0$. Otherwise, the rival bidder with valuation $\underline{v}$ should not bid zero, but rather marginally above zero, in order to dramatically increase the probability that he wins.

deviate, since lowering the bid marginally would not reduce the probability (which is one) that he wins. Similarly, neither bidder can bid $\bar{b}$ with a strictly positive probability in equilibrium. Otherwise, it would be worthwhile for the other bidder to bid slightly above $\bar{b}$ to significantly increase the probability that he wins. The first part of the argument holds only for the special case of two bidders. With three or more bidders, it is possible that not all bidders have the same highest possible bid. See Parreiras and Rubinchik (2006) for more details.

In equilibrium, bidder i, $i = 1, 2$, with valuation v is faced with the problem of selecting the bid, b, that will maximize his expected payoff,

$$\max_b vF_j(\varphi_j(b)) - b, \tag{1}$$

where $j \neq i$. By bidding b, bidder i wins with probability $F_j(\varphi_j(b))$ and since the auction is an all-pay auction the bid must be paid regardless of whether the bidder wins or loses.[9]

Assuming a bid in the interior is optimal, the first order condition is

$$v\frac{dF_j(\varphi_j(b))}{db} - 1 = 0.$$

If bidder i with valuation v bids b in equilibrium, it follows that $v = \varphi_i(b)$, and we can write the first order condition as

$$\frac{dF_j(\varphi_j(b))}{db} = \frac{1}{\varphi_i(b)}, \tag{2}$$

or, alternatively, as

$$\frac{d\varphi_j(b)}{db} = \frac{1}{\varphi_i(b)f_j(\varphi_j(b))} \tag{3}$$

Either way of writing the first order condition can be useful, depending on which aspect of the auction is being examined. (2) details how much an increase in bidder i's bid would increase his probability of winning, $F_j(\varphi_j(b))$, whereas (3) reveals how many more types of bidder j he would outbid by doing so.

Keeping in mind that the two bidders must share the same maximal bid, the system of differential equations described by the two bidders' first

[9] As pointed out by Parreiras and Rubinchik (2006, footnote 1), the assumption that the cost of bidding is linear in the bid, b, is without loss of generality.

order conditions can be analyzed to uncover the consequences of a change in the bidders' beliefs. In particular, we will examine what happens when one bidder's distribution is *truncated*. Specifically, assume it becomes known before the auction that bidder 1's valuation is not above some $\widehat{v}_1$, $\widehat{v}_1 \in (\underline{v}, \overline{v}_1)$. In other words, bidder 1 is revealed to be not too strong, which will make the other bidder more optimistic about his chances in the auction, other things being equal. Initially, we assume $\widehat{v}_1$ is exogenous but we later turn to a larger game in which $\widehat{v}_1$ is determined endogenously.

Let

$$G_1(v) = \frac{F_1(v)}{F_1(\widehat{v}_1)},\ v \in [\underline{v}, \widehat{v}_1]$$

denote the new, updated beliefs concerning bidder 1, and let γ_1 and γ_2 denote the new inverse bidding functions. $G_1(\gamma_1(b))$ then describes the distribution of bidder 1's bids, after the truncation becomes known and bidders adjust bidding strategies to reflect the change in the competitive environment. As a consequence of the truncation, the common maximal bid is also likely to change, from $\overline{b}_\varphi$ to $\overline{b}_\gamma$. In the following, we let $\widehat{b}_\varphi$ denote the bid that bidder 1 would have submitted in the original game (before the truncation), had his type been exactly $\widehat{v}_1$.

In the next section we engage in a comparative statics exercise, analyzing how the equilibrium depends on the truncation, $\widehat{v}_1$. Notice that $\widehat{v}_1$ can be viewed as a measure of the "strength" of bidder 1, or how willing bidder 1 at most is to fight for the object in a contest.

In Section 4 we consider a game in which the truncation arises endogenously.

3 Comparative Statics

We begin by examining how the truncation affects the relative bids of the two bidders, and on how the decision to participate or stay out of the auction is impacted. Thereafter we discuss how individual bids change, as well as how payoff and total expenditure depends on the truncation.

3.1 Pivotal rival types

In this section we follow the approach in Amann and Leininger (1996) to investigate how the truncation affects who the bidder's *pivotal rival type* is.

Bidder i's pivotal rival type refers to the type of bidder j that matches bidder i's bid. If bidder j's type is below (above) this critical type, bidder i will win (lose) the auction.

Should the pivotal types change, the reason must be that the *relative bids* or relative aggressiveness of the two bidders have changed. Later, we will examine how the *absolute bids* for any given bidder is impacted by the truncation.

When bids are in the interior, the first order conditions can be written as

$$\frac{d\gamma_1(b)}{db} = \frac{1}{\gamma_2(b)} \frac{F_1(\widehat{v}_1)}{f_1(\gamma_1(b))} \tag{4}$$

$$\frac{d\gamma_2(b)}{db} = \frac{1}{\gamma_1(b)} \frac{1}{f_2(\gamma_2(b))}, \tag{5}$$

which is an autonomous system to which we can apply the following standard solution procedure. Dividing the two conditions yields

$$\frac{d\gamma_1(b)}{d\gamma_2(b)} = F_1(\widehat{v}_1) \frac{\gamma_1(b) f_2(\gamma_2(b))}{\gamma_2(b) f_1(\gamma_1(b))}. \tag{6}$$

The latter suggests it is worthwhile thinking of the problem in terms of a relationship between the types of the two bidders who would submit the same bid, and therefore tie in the auction – the pivotal types. If bidder 2's type is v, we define $k(v|\widehat{v}_1)$ as the type of bidder 1 who bidder 2 would tie with, as in Amann and Leininger (1996). Recall the requirement that $k(\overline{v}_2|\widehat{v}_1) = \widehat{v}_1$ since the two bidders share a common maximal bid.

$k(\cdot|\cdot)$ is defined for all $v \in [v_2^0, \overline{v}_2]$, i.e. for all types who participates in the auction. Define

$$P_2 = (\max\{v_2^0(\widehat{v}_1), v_2^0(\overline{v}_1)\}, \overline{v}_2],$$

where $v_2^0(\widehat{v}_1)$ is the infimum of those of bidder 2's types who participate in the auction, given the truncation is at $\widehat{v}_1$. Hence, P_2 is the set of bidder 2's types who submit strictly positive bids both before and after the truncation. In a similar manner we also define

$$P_1 = (\max\{v_1^0(\widehat{v}_1), v_1^0(\overline{v}_1)\}, \widehat{v}_1].$$

When there is no risk of confusion, we suppress the dependence on $\widehat{v}_1$ and simply write $k(v)$. Given the definition of $k(v)$, (6) can be written as

$$\frac{dk(v)}{dv} = F_1(\widehat{v}_1) \frac{k(v) f_2(v)}{v f_1(k(v))}, \tag{7}$$

and our first result follows immediately.[10]

Lemma 1 *$k(\overline{v}_2|\widehat{v}_1) < k(\overline{v}_2|\overline{v}_1)$, and $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect at most once on P_2.*

Proof. The first part follows from $k(\overline{v}_2|\widehat{v}_1) = \widehat{v}_1 < \overline{v}_1 = k(\overline{v}_2|\overline{v}_1)$. There is at most one intersection on P_2 because if $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ touch for any $v \in P_2$, (7) implies that the former is strictly flatter than the latter. ■

Lemma 1 implies that either $k(v|\widehat{v}_1) < k(v|\overline{v}_1)$ for all $v \in P_2$ (no intersection), or that $k(v|\widehat{v}_1) > k(v|\overline{v}_1)$ if v is low and $k(v|\widehat{v}_1) < k(v|\overline{v}_1)$ if v is high (one intersection). Which case applies depends on how the bidders' propensities to participate in the auction (summarized by v_1^0 and v_2^0) change as a consequence of the truncation.

Remembering that a change in k reveals a change in the relative aggressiveness of the two bidders, we notice that when k falls after the truncation bidder 2 becomes less aggressive compared to bidder 1, i.e. he outbids fewer types. Equivalently, bidder 1 becomes relatively more aggressive, possibly in an effort to compensate for the fact that he is now considered weaker.

When k increases after the truncation, which may occur for low types, bidder 2 is the bidder who becomes more aggressive. Indeed, the set of types for which he participates in the auction may grow. Intuitively, this increased aggressiveness can occur if the update in information makes him more optimistic about his chances in the auction, and thus more willing to compete.

Since bidding strategies and payoffs are determined by $k(\cdot)$ (see below), we devote the remainder of the subsection to understanding under what circumstances $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ will intersect.

Returning to the analysis of (7), which is a separable equation, the solution must satisfy

$$\int_{\underline{v}}^{k(v)} \frac{f_1(x)}{x} dx = F_1(\widehat{v}_1) \int_{\underline{v}}^{v} \frac{f_2(x)}{x} dx + c(\widehat{v}_1), \tag{8}$$

where $c(\widehat{v}_1)$ is a constant that ensures $k(\overline{v}_2) = \widehat{v}_1$. Hence, $c(\widehat{v}_1)$ is

$$c(\widehat{v}_1) = \int_{\underline{v}}^{\widehat{v}_1} \frac{f_1(x)}{x} dx - F_1(\widehat{v}_1) \int_{\underline{v}}^{\overline{v}_2} \frac{f_2(x)}{x} dx \tag{9}$$

[10] As noticed by Amann and Leininger (1996), given the boundary condition that $k(\overline{v}_2|\widehat{v}_1) = \widehat{v}_1$, $k(v|\widehat{v}_1)$ is uniquely determined.

and it follows that $k(v)$ is implicitly and uniquely defined by

$$\int_{k(v)}^{\widehat{v}_1} \frac{f_1(x)}{x} dx = F_1(\widehat{v}_1) \int_{v}^{\overline{v}_2} \frac{f_2(x)}{x} dx. \tag{10}$$

To recapitulate, (7) was solved by using the boundary condition that $k(\overline{v}_2) = \widehat{v}_1$. With this boundary condition, (7) describes how $k(v)$ changes as v moves leftward. However, as we "shoot" backwards, it is possible that $k(v)$ intersects one of the axes with origin $(\underline{v}, \underline{v})$, rather than terminating at $(\underline{v}, \underline{v})$. This allows us to determine v_1^0 and v_2^0.

For instance, if $k(\underline{v}) > \underline{v}$ the implication is that bidder 1 does not participate if his type is in the interval $[\underline{v}, v_1^0)$, where $v_1^0 = k(\underline{v})$. On the other hand, if $k^{-1}(\underline{v}) > \underline{v}$, bidder 2 stays out for a mass of types, $[\underline{v}, v_2^0)$, with $k(v_2^0) = \underline{v}$.

Both bidders participate in the auction with probability one only if $k(\underline{v}) = \underline{v}$. There are two separate cases with this feature. If the two terms in (9) are finite but of the same size (implying $c(\widehat{v}_1) = 0$), then $k(\underline{v}) = \underline{v}$ clearly satisfies (8) or (10). $k(\underline{v}) = \underline{v}$ must also hold if both terms in (9) are infinite, which necessitates $\underline{v} = 0$.

Bidder 1 stays out with positive probability if $c(\widehat{v}_1) > 0$, whereas bidder 2 stays out with positive probability if $c(\widehat{v}_1) < 0$.

In general, the sign and size of $c(\overline{v}_1)$ plays an important role in determining how the pivotal rival types compare before and after the information update. We first consider the simpler case where bidder 2 stays out of the auction with positive probability before the truncation ($c(\overline{v}_1) < 0$, or $v_2^0(\overline{v}_1) > \underline{v}$). After he learns his competitor is weaker, he will be more willing to participate.

Proposition 1 *Assume $v_2^0(\overline{v}_1) > \underline{v}$. Then:*

1. *$v_2^0(\widehat{v}_1) < v_2^0(\overline{v}_1)$. That is, bidder 2 participates more often after he learns the competitor is weaker than initially thought.*

2. *$k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect on P_2.*

Proof. Evaluated at v_2^0, (10) becomes

$$\int_{\underline{v}}^{\widehat{v}_1} \frac{f_1(x)}{x} dx = F_1(\widehat{v}_1) \int_{v_2^0}^{\overline{v}_2} \frac{f_2(x)}{x} dx, \tag{11}$$

since $k(v_2^0) = \underline{v}$. Implicit differentiation of (11) yields

$$\frac{dv_2^0}{d\widehat{v}_1} = \frac{f_1(\widehat{v}_1)}{F_1(\widehat{v}_1)^2} \frac{v_2^0}{f_2(v_2^0)} \int_{\underline{v}}^{\widehat{v}_1} \left(\frac{1}{x} - \frac{1}{\widehat{v}_1} \right) f_1(x) dx > 0, \tag{12}$$

which completes the proof of the first part. Lemma 1 and the first part of the proposition imply the second part. ■

In contrast to the unambiguous conclusion in Proposition 1, the truncation can have various consequences when bidder 1 initially stays out of the auction with positive probability.

Proposition 2 *Assume $v_1^0(\overline{v}_1) > \underline{v}$. Then:*

1. *$v_1^0(\widehat{v}_1) > \underline{v}$ for any $\widehat{v}_1 \in (\underline{v}, \overline{v}_1)$.*
2. *There exists a unique number, $w \in (\underline{v}, \overline{v}_2)$, possibly larger than $\overline{v}_1$, such that if $\widehat{v}_1 > w$ then $v_1^0(\widehat{v}_1) > v_1^0(\overline{v}_1)$ but if $\widehat{v}_1 < w$ then $v_1^0(\widehat{v}_1) < v_1^0(\overline{v}_1)$.*
3. *If $v_1^0(\widehat{v}_1) > v_1^0(\overline{v}_1)$ then $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect on $v \in (\underline{v}, \overline{v}_2)$. If $v_1^0(\widehat{v}_1) < v_1^0(\overline{v}_1)$ then $k(v|\widehat{v}_1) < k(v|\overline{v}_1)$ for all $v \in (\underline{v}, \overline{v}_2]$.*

Finally, if $v_1^0(\overline{v}_1) = \underline{v}$ and both terms in (9) are finite, then $v_1^0(\widehat{v}_1) > \underline{v}$ for any $\widehat{v}_1 \in (\underline{v}, \overline{v}_1)$. Moreover, $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect on $v \in (\underline{v}, \overline{v}_2)$.

Proof. $v_1^0(\widehat{v}_1) > \underline{v}$ if and only if $c(\widehat{v}_1) > 0$, or if

$$C(\widehat{v}_1) = \frac{1}{F_1(\widehat{v}_1)} \int_{\underline{v}}^{\widehat{v}_1} \frac{f_1(x)}{x} dx - \int_{\underline{v}}^{\overline{v}_2} \frac{f_2(x)}{x} dx > 0. \tag{13}$$

If the first term is infinite it will remain infinite when $\widehat{v}_1$ changes. Otherwise, $C(\widehat{v}_1)$ is strictly decreasing in $\widehat{v}_1$ and so in either case we conclude that if $c(\overline{v}_1) > 0$ then $c(\widehat{v}_1) > 0$ as well. Hence, after the truncation, bidder 1 will still stay out with strictly positive probability, which proves the first part.

Evaluated at $\underline{v}$ (10) is

$$\int_{v_1^0}^{\widehat{v}_1} \frac{f_1(x)}{x} dx = F_1(\widehat{v}_1) \int_{\underline{v}}^{\overline{v}_2} \frac{f_2(x)}{x} dx, \tag{14}$$

since $k(\underline{v}) = v_1^0$, by assumption. Implicit differentiation of (14) produces

$$\frac{dv_1^0}{d\widehat{v}_1} = v_1^0 \frac{f_1(\widehat{v}_1)}{f_1(v_1^0)} \int_{\underline{v}}^{\overline{v}_2} \left(\frac{1}{\widehat{v}_1} - \frac{1}{x} \right) f_2(x) dx.$$

The sign is determined by the last term, which is strictly positive if $\widehat{v}_1 = \underline{v}$ and strictly negative if $\widehat{v}_1 = \overline{v}_2$. Since this term is also strictly decreasing in $\widehat{v}_1$, there is a unique value of $\widehat{v}_1$, v', between $\underline{v}$ and $\overline{v}_2$ where the term is zero. If $\overline{v}_1 < v'$ then $v_1^0(\widehat{v}_1)$ is strictly increasing in $\widehat{v}_1$. Otherwise, $v_1^0(\widehat{v}_1)$ is hump-shaped in $\widehat{v}_1$. In either case, $v_1^0(\widehat{v}_1)$ is strictly below the 45° line (since type $\widehat{v}_1$ bids $\overline{b} > \underline{v}$), with $v_1^0(\overline{v}_1) > \underline{v}$ (by assumption) and $v_1^0(\underline{v}) = \underline{v}$. The second part of the Proposition follows. The third part is proven by applying Lemma 1. The proof of the final part of the Proposition mirrors that of the first part. ■

Notice that Proposition 2 implies that if bidder 2 participates with probability one before the truncation he will continue to do so after.

For bidder 1 Proposition 2 means that he stays out for fewer types after the truncation if $\overline{v}_1$ is small compared to $\overline{v}_2$ or if the truncation is very large. Otherwise, bidder 1 will stay out for more types than before the truncation.

In the first case, bidder 2 becomes so disdainful of the competition that he lowers his bid, thereby making it more profitable for bidder 1 to participate in the contest.[11] In the second case, in contrast, bidder 2 seizes upon bidder 1's diminished strength and increases his own level of aggressiveness, at least if his type is low. This makes it less profitable to participate for bidder 1 with a low type, and, demoralized, he thus stays out of the auction for more types.

Example 1 illustrates Propositions 1 and 2.

Example 1: Assume that

$$\begin{aligned} F_1(v) &= \left(\frac{v}{\overline{v}_1}\right)^5, \ v \in [0, \overline{v}_1], \\ F_2(v) &= v^2, \ v \in [0,1]. \end{aligned}$$

A special case of this specification, with $\overline{v}_1 = 1$, is a leading example in Amann and Leininger (1996). Using (9), with $\overline{v}_1$ in place of $\widehat{v}_1$, it can easily be shown that $c(\overline{v}_1)$ is negative if $\overline{v}_1 > \frac{5}{8}$, positive if $\overline{v}_1 < \frac{5}{8}$. Hence, if $\overline{v}_1$ is small, bidder 1 stays out for a mass of types. Using (14), it can be verified that

$$v_1^0(\overline{v}_1) = \overline{v}_1 \left(1 - \frac{8}{5}\overline{v}_1\right)^{\frac{1}{4}}, \tag{15}$$

[11] In Section 3.2 we verify that bidder 2 lowers his bid, and that even though bidder 1 participates for more *types*, the *probability* that he participates, $1 - \frac{F_1(v_1^0)}{F_1(\widehat{v}_1)}$, must decline.

which is depicted in Figure 1(a). If $\overline{v}_1$ is large, bidder 2 stays out for a mass of types. Using (11) we find

$$v_2^0(\overline{v}_1) = 1 - \frac{5}{8}\frac{1}{\overline{v}_1},$$

which is shown in Figure 1(b).

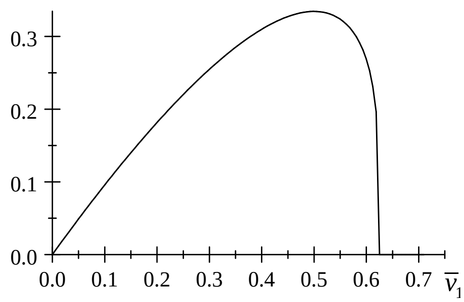

Figure 1(a): $v_1^0(\overline{v}_1)$

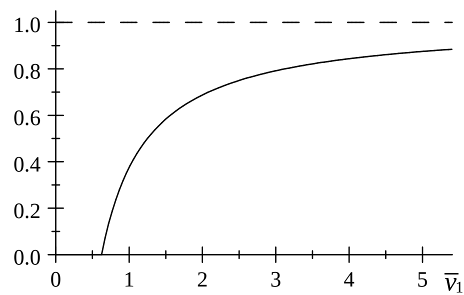

Figure 1(b): $v_2^0(\overline{v}_1)$

The example verifies the conclusions in Propositions 1 and 2. Starting with large values of $\overline{v}_1$, we notice that bidder 2 stays out with a large probability, but that this probability diminishes and eventually disappears as $\overline{v}_1$ decreases (as a result of the truncation). Indeed, when $\overline{v}_1$ becomes sufficiently small, bidder 1 begins staying out with positive probability. As $\overline{v}_1$ is reduced more, bidder 1 stays out for more types, but this trend reverses itself as $\overline{v}_1$ becomes even smaller. Assuming $k(v) \geq 0$, (10) can be used to derive

$$k(v) = \overline{v}_1 \left(1 - \frac{8}{5}\overline{v}_1(1-v)\right)^{\frac{1}{4}}$$

and Figure 2 shows how it depends on $\overline{v}_1$. $\square$

Jointly, Propositions 1 and 2 cover all but one possibility, namely that $v_1^0(\overline{v}_1) = v_2^0(\overline{v}_1) = \underline{v}$ because both terms in (9) are infinite, such that (10) is satisfied at $v = k = \underline{v}$. This can occur only if $\underline{v} = 0$. On the other hand, if $\underline{v} = 0$ both integrals in (9) are divergent if the densities are bounded away from zero (as in example 2, to follow, but contrary to example 1).

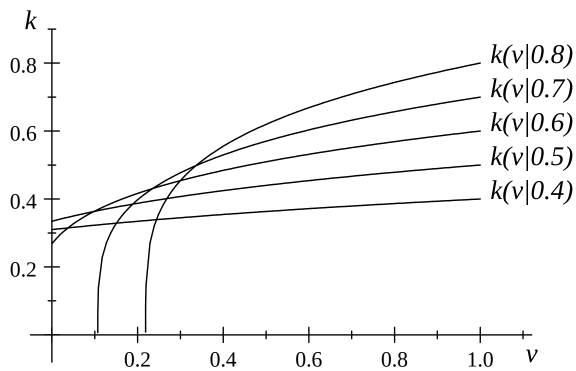

Figure 2: $k(v|\overline{v}_1)$ for $\overline{v}_1 = 0.4, 0.5, 0.6, 0.7, 0.8$.

In this case, a reduction in $\widehat{v}_1$ will leave each term in $c(\widehat{v}_1)$ unchanged, at infinity, and (10) remains satisfied at $v = k = \underline{v}$.[12] Hence, both bidders will participate with probability one, both before and after the truncation. We state this result formally below, as a counterpart to the final part of Proposition 2.

Proposition 3 *Assume $\underline{v} = 0$. If $v_1^0(\overline{v}_1) = v_2^0(\overline{v}_1) = 0$ and both terms in (9) are infinite, then $v_1^0(\widehat{v}_1) = v_2^0(\widehat{v}_1) = 0$ for any $\widehat{v}_1 \in (0, \overline{v}_1)$. Moreover, $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect.*

Proof. The first part was proven in the text. For the second part, at any $v > 0$ implicit differentiation of (10) yields

$$\frac{dk(v|\widehat{v}_1)}{d\widehat{v}_1} = \frac{k(v|\widehat{v}_1)f_1(\widehat{v}_1)}{f_1(k(v|\widehat{v}_1))}\left[\frac{1}{\widehat{v}_1} - \int_v^{\overline{v}_2} \frac{f_2(x)}{x}dx\right].$$

By assumption, the term in brackets goes to $-\infty$ as $v \to 0$, while it becomes positive as v approaches $\overline{v}_2$. Hence, when $\widehat{v}_1$ decreases $k(v|\widehat{v}_1)$ increases for small values of v, and decreases for large values of v (with a unique point of crossing). ■

Having considered all the possible cases, we observe that if $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect it must necessarily be because $v_1^0(\overline{v}_1) > \underline{v}$. Hence, $v_2^0 = \underline{v}$ before and after the truncation, implying that $P_2 = (\underline{v}, \overline{v}_2]$.

[12] If the integrals are divergent they remain divergent after the truncation.

3.2 Bids, distributions of bids, and interim expected payoff

We start by deriving bidder 2's bid. Following Amann and Leininger (1996) once again, (5) and the inverse function theorem combine to yield

$$\frac{db}{d\gamma_2} = \gamma_1(b) f_2(\gamma_2(b)). \tag{16}$$

Since $\gamma_2(b)$ simply identifies the type of bidder 2 who bids b and $\gamma_1(b)$ is the pivotal type of bidder 1 with whom bidder 2 ties, (16) can be written as

$$b_2'(v) = k(v) f_2(v). \tag{17}$$

Knowing that $b_2(v_2^0) = 0$, it follows that

$$b_2(v) = \int_{v_2^0}^{v} k(x) f_2(x) dx, \tag{18}$$

if $v \in [v_2^0, \overline{v}_2]$. If $v < v_2^0$, bidder 2 stays out of the auction. As suggested by Amann and Leininger (1996), bidder 1's bid can then be found by inverting $k(v)$ and remembering that bidder 2 with valuation v bids the same as bidder 1 with valuation $k(v)$, or $b_1(v) = b_2(k^{-1}(v))$. Alternatively, we can repeat the above process to find that

$$b_1(v) = \int_{v_1^0}^{v} k^{-1}(x) \frac{f_1(x)}{F_1(\widehat{v}_1)} dx, \tag{19}$$

if $v \in [v_1^0, \widehat{v}_1]$. If $v < v_1^0$, bidder 1 does not participate. Since we have identified how k depends on $\widehat{v}_1$, (18) and (19) allow us to infer how the bidding strategies depend on $\widehat{v}_1$ as well.

First, if $k(v|\widehat{v}_1)$ is globally below $k(v|\overline{v}_1)$, bidder 2 clearly lowers his bid *regardless* of his valuation after the truncation. Similarly, bidder 1 increases his bid for all types, since k^{-1} has increased. This is obviously consistent with the fact that bidder 2's relative aggressiveness declines after the truncation, as is evidenced by $k(v|\widehat{v}_1)$ being smaller than $k(v|\overline{v}_1)$. Bidder 2's lower bid is also in keeping with the decrease in v_1^0 implied by the ranking of $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$. In particular, bidder 1's increased participation (decrease in v_1^0) can be rationalized only if bidder 2 is less aggressive for low types, which would improve the profitability of participation.

Second, when $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect, b_2 increases when v is small since k increases when v is small (and v_2^0 does not increase). This also explains why v_1^0 may increase. If bidder 2 becomes more aggressive, it might deter more of bidder 1's types. We can also infer from (18) that if there is a type for which bidder 2 bids the same before and after the truncation, then he will bid strictly less for higher types after the truncation, since k decreases for large values of v. In similar fashion, (19) leads to the conclusion that if bidder 1 has a type for which he bids the same before and after the truncation, then he will increase his bid after the truncation above this cut-off type. In other words, for the set of types where bids are positive, old and new bidding strategies will cross at most once for either bidder.

Proposition 4 *If $k(v|\widehat{v}_1) < k(v|\overline{v}_1)$ for all $v \in P_2 = (\underline{v}, \overline{v}_2]$ then bidder 2 bids less aggressively and bidder 1 more aggressively after the truncation, or $b_2(v|\widehat{v}_1) < b_2(v|\overline{v}_1)$ for all $v \in P_2 = (\underline{v}, \overline{v}_2]$ and $b_1(v|\widehat{v}_1) > b_1(v|\overline{v}_1)$ for all $v \in P_1$. Otherwise, there is a unique type in P_2 such that bidder 2 bids more aggressively below that type and less aggressively above that type, while there is another unique type in P_1 for which the opposite holds for bidder 1. Moreover, the common maximal bid, $\overline{b}_\gamma$, falls in the interval $(\widehat{b}_\varphi, \overline{b}_\varphi)$.*

Proof. The first part follows directly from (18) and (19), which also imply that $\overline{b}_\gamma \in (\widehat{b}_\varphi, \overline{b}_\varphi)$. The second part follows from the argument prior to the Proposition if it can be established that $\overline{b}_\gamma$ also falls in the interval $(\widehat{b}_\varphi, \overline{b}_\varphi)$ when $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect. If $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ cross (18) and (19) imply that bidder 2 bids more aggressively if his type is low, whereas bidder 1 bids less aggressively if his type is low. Consider now the point of intersection, v', between $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$. At this point, bidder 2 bids more aggressively, by (18). However, the pivotal rival type is unchanged, at $k(v')$, meaning that bidder 1 must also increase his bid if he happens to have this type in order to still tie with v'. Hence, $b_1(v|\widehat{v}_1)$ must cross $b_1(v|\overline{v}_1)$, and so $\overline{b}_\gamma > \widehat{b}_\varphi$. At the point where they cross, it must be the case that $\gamma_1(b) = \varphi_1(b)$ (the same bid is submitted by the same type), and so $G_1(\gamma_1(b)) > F_1(\varphi_1(b))$. If bidder 2 bids more aggressively than before for all types, we would have $\gamma_2(b) < \varphi_2(b)$ for all $b \in (0, \overline{b}_\varphi)$, meaning that $G_1(\gamma_1(b))$ would be steeper than $F_1(\varphi_1(b))$. Hence, $G_1(\gamma_1(b))$ would equal 1 to the left of the point where $F_1(\varphi_1(b))$ equals one, the meaning of which is that $\overline{b}_\gamma < \overline{b}_\varphi$. However, since $b_2(\overline{v}_2|\widehat{v}_1) = \overline{b}_\gamma$ this contradicts the assumption that bidder 2 bids more aggressively for all types. Hence, bidder 2 must bid less aggressively at the top, $\bar{b}_\gamma < \bar{b}_\varphi$, and the proof is concluded. ■

Some intuition for why $\bar{b}_\gamma > \widehat{b}_\varphi$ may be obtained by contemplating the deviations that would be profitable if bidders for some reason maintained the original strategies after the truncation. Since $G_1(\varphi_1(b))$ is steeper than $F_1(\varphi_1(b))$ on $b \in [0, \widehat{b}_\varphi]$ the return for bidder 2 of increasing his bid increases after the truncation, and bidder 2 would as a consequence decide to bid $\widehat{b}_\varphi$ for a mass of types. However, this would in turn give bidder 1 an incentive to bid marginally above $\widehat{b}_\varphi$, if his type is sufficiently high (close to $\widehat{v}_1$), in order to win with a much higher probability. This effect puts upwards pressure on the highest possible bid after the truncation.

Thus, we have seen that bidder 1 bids more aggressively for at least a subset of types after he has been revealed to be weaker. However, this increased aggressiveness is counteracted by the fact that he is more likely to have a low type.

Proposition 5 *$F_1(\varphi_1(b))$ first order stochastically dominates $G_1(\gamma_1(b))$, i.e. $F_1(\varphi_1(0)) \leq G_1(\gamma_1(0))$ and $F_1(\varphi_1(b)) < G_1(\gamma_1(b))$ for all $b \in (0, \bar{b}_\gamma]$.*

Proof. The fact that $\bar{b}_\gamma < \bar{b}_\varphi$ means that $G_1(\gamma_1(\bar{b}_\gamma)) = 1 > F_1(\varphi_1(\bar{b}_\gamma))$. If $G_1(\gamma_1)$ and $F_1(\varphi_1)$ cross as b is reduced, the former must therefore be at least as steep as the latter at the point of intersection closest to $\bar{b}_\gamma$. This necessitates $\gamma_2 \leq \varphi_2$ or $F_2(\gamma_2) \leq F_2(\varphi_2)$. Since $G_1(\gamma_1) = F_1(\varphi_1)$, it must also be the case that $\gamma_1 < \varphi_1$, which means that $F_2(\gamma_2)$ is strictly steeper than $F_2(\varphi_2)$. As we move leftward, these properties combine to ensure that $F_2(\gamma_2)$ is below $F_2(\varphi_2)$ and steeper, and $G_1(\gamma_1)$ is below $F_1(\varphi_1)$ and steeper. Hence, either $F_2(\gamma_2)$ or $G_1(\gamma_1)$ must intersect the horizontal axis for some $b > 0$. However, this contradicts the equilibrium property that the smallest bid is zero. ■

Although Proposition 2 shows that bidder 1 may participate for fewer (or more) types after the truncation, Proposition 5 proves that the *probability* that bidder 1 stays out of the auction can not decrease when bidder 1 becomes "weaker". This is clearly illustrated in Example 1, where the probability that bidder 1 stays inactive is $(v_1^0/\overline{v}_1)^5$, which either increases when $\overline{v}_1$ decreases, given (15), or remains constant at zero. We summarize the findings concerning the probability of participation in the following Corollary.

Corollary 1 *After the truncation the probability that bidder 1 stays out* *(weakly) increases, while the probability that bidder 2 stays out (weakly) decreases.*

Proof. The first part reiterates Proposition 5, while the second part is a trivial consequence of the first part of Propositions 1 and 2. ■

Proposition 5 also implies that bidder 2 is better off after the truncation, simply because he faces a "weaker" distribution of bids.

Corollary 2 *For any type, v, where $b_2(v|\widehat{v}_1) > 0$, bidder 2 is strictly better off after the truncation.*

Proof. If bidder 2's type is such that he submitted a positive bid before the truncation, he can ensure himself a strictly higher payoff by not changing his bid, by Proposition 5. If he did not participate before the truncation but submits a strictly positive bid after, his payoff must necessarily have changed from zero to a strictly positive value. ■

Proposition 5 also means that the expected payment from bidder 1 declines. That is, the seller earns less on bidder 1 after the truncation.

Recall that there are circumstances under which bidder 2 bids less aggressively for all types after the truncation. In this case, bidder 1 is better off as well (if he participates) and the seller earns less on both bidders.

However, there are other cases in which bidder 2 reacts to the truncation by bidding more aggressively if he has a low type. In this case, the above conclusions do not necessarily hold, as illustrated in example 2.

Example 2: Assume that both bidders' types are drawn from uniform distributions,

$$F_i(v) = \frac{v}{\overline{v}_i},\ v \in [0, \overline{v}_i],$$

and normalize $\overline{v}_2 = 1$. Clark and Riis (2000) and Sahuguet (2006) examine modified all-pay auctions with the assumption that both distributions are uniform. Since both terms in (9) are infinite, both bidders always participate. From (7) or (10),

$$k(v|\overline{v}_1) = \overline{v}_1 v^{\overline{v}_1},$$

and

$$k^{-1}(v|\overline{v}_1) = \left(\frac{v}{\overline{v}_1}\right)^{\frac{1}{\overline{v}_1}}. \tag{20}$$

Bids are therefore

$$\begin{aligned} b_1(v|\overline{v}_1) &= \frac{\overline{v}_1}{\overline{v}_1+1}\left(\frac{v}{\overline{v}_1}\right)^{\frac{\overline{v}_1+1}{\overline{v}_1}} \\ b_2(v|\overline{v}_1) &= \frac{\overline{v}_1}{\overline{v}_1+1}v^{\overline{v}_1+1}, \end{aligned}$$

with $\overline{b}_\varphi = \frac{\overline{v}_1}{\overline{v}_1+1}$. Consider bidder 1 with type v. Before the truncation, he would have won if bidder 2's valuation was below $k^{-1}(v|\overline{v}_1)$, which occurs with probability $k^{-1}(v|\overline{v}_1)$, given the uniform distribution. Hence, his expected payoff is

$$\begin{aligned} EU_1(v|\overline{v}_1) &= v\left(\frac{v}{\overline{v}_1}\right)^{\frac{1}{\overline{v}_1}} - \frac{\overline{v}_1}{\overline{v}_1+1}\left(\frac{v}{\overline{v}_1}\right)^{\frac{\overline{v}_1+1}{\overline{v}_1}} \\ &= \frac{\overline{v}_1}{\overline{v}_1+1}\overline{v}_1^{-\frac{1}{\overline{v}_1}} v^{\frac{\overline{v}_1+1}{\overline{v}_1}}. \end{aligned}$$

To illustrate, assume that $\overline{v}_1 > v = 2$, but that the truncation is to $\widehat{v}_1 = v = 2$. In this case, $EU_1(2|\overline{v}_1)$ is U-shaped in $\overline{v}_1$, with the post-truncation payoff at the point where $\overline{v}_1 = 2$. Hence, if $\overline{v}_1$ is relatively small initially (between 2 and approximately 3.7), bidder 1 with type $v = 2$ is made better off after the truncation, whereas he is made worse off when $\overline{v}_1$ is large to begin with (above 3.7).

Figure 3 depicts bidder 2's bidding strategy, $b_2(v|\overline{v}_1)$, for different values of $\overline{v}_1$. Notice that the stronger bidder 1 is perceived to be (the higher $\overline{v}_1$ is), the more carefully bidder 2 bids if his type is not too high, but the strategies do cross for high values of v.

From bidder 1's perspective, the curve describing bidder 2's strategy is essentially a feasible set. Any point on the curve is a (v, b) pair that reveals the type, v, that bidder 1 would tie with if he bids b. Since bidder 2's distribution is uniform on $[0, 1]$ the horizontal axis also measures the probability that bidder 1 wins by bidding b. Hence, the figure shows the trade-off between winning more often and paying more, and it illustrates that this trade-off changes when $\overline{v}_1$ changes. To compare, the figure therefore also contains the indifference curve on which bidder 1 with type $v = 2$ maximizes his expected payoff when $\overline{v}_1 = 2$ (after the truncation). Since utility is increasing to the south-east, there is a feasible point that is better for bidder 1 with type $v = 2$ when $\overline{v}_1 = 5$ than when $\overline{v}_1 = 2$, but this is not the case when $\overline{v}_1 = 3$.

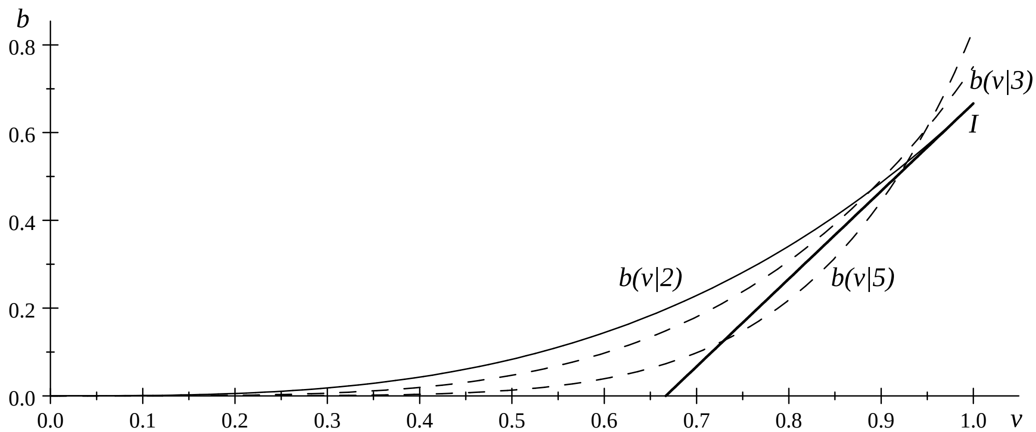

Figure 3: Bidder 2's strategy, $b(v|\overline{v}_1)$, for $\overline{v}_1 = 2, 3, 5$, and indifference curve, I, for bidder 1 with $v = 2$.

The expected payment from the bidders are, as a function of $\overline{v}_1$,

$$EP_1(\overline{v}_1) = \int_0^{\overline{v}_1} b_1(v|\overline{v}_1)\frac{1}{\overline{v}_1}dv = \frac{\overline{v}_1^2}{3\overline{v}_1 + 2\overline{v}_1^2 + 1} \quad (21)$$

$$EP_2(\overline{v}_1) = \int_0^1 b_2(v|\overline{v}_1)dv = \frac{\overline{v}_1}{3\overline{v}_1 + \overline{v}_1^2 + 2}, \quad (22)$$

and they are shown in Figure 4. The figure verifies the implication of Proposition 5, namely that expected payment from bidder 1 declines after the truncation (which corresponds to decreasing $\overline{v}_1$). However, bidder 2's expected payment is a non-monotonic function of $\overline{v}_1$. When the bidders are not too different, i.e. when $\overline{v}_1$ is close to one, bidder 2's expected payment is large. However, if the asymmetry is large, or $\overline{v}_1$ is far removed from 1, expected payment suffers. Summing the expected payment from the two bidders, we see that it is hump-shaped, with a peak at approximately $\overline{v}_1 = 12.3$. Hence, if $\overline{v}_1 < 12.3$, any truncation will decrease total expected payment, but if $\overline{v}_1$ is larger, a small truncation or decrease in $\overline{v}_1$ will in fact increase total revenue. In the latter case, the increased payment from bidder 2 more than makes up for the small decrease in bidder 1's payment. $\square$

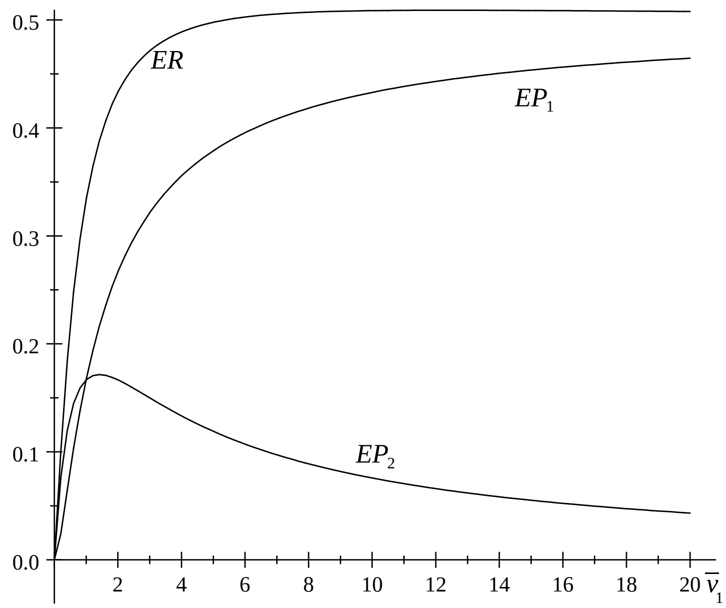

Figure 4: Expected payment from each bidder and total expected payment, ER, as a function of $\overline{v}_1$.

Example 2 and the discussion leading up to it yield the following results.

Proposition 6 *If $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect, bidder 1 is strictly better off after the truncation for any type where $b_1(v|\widehat{v}_1) > 0$. If $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect, bidder 1 is either strictly worse off for all types with $b_1(v|\overline{v}_1) > 0$, or there is a unique type in $(\underline{v}, \widehat{v}_1)$ such that he is worse off below that type and strictly better off above that type.*

Proof. Using the Envelope Theorem on (1) proves that the expected payoff to bidder 1 changes with v in the following way,

$$\frac{dEU_1(v|\overline{v}_1)}{dv} = F_2(\varphi_2(b)) = F_2(k^{-1}(v|\overline{v}_1)), \tag{23}$$

where the second equality comes from the fact that bidder 1 with type v bids the same as bidder 2 with type $k^{-1}(v|\overline{v}_1)$. Hence, expected payoff can be written as

$$EU_1(v|\overline{v}_1) = EU_1(v_1^0(\overline{v}_1)|\overline{v}_1) + \int_{v_1^0(\overline{v}_1)}^{v} F_2(k^{-1}(x|\overline{v}_1))dx, \tag{24}$$

when $v \in [v_1^0(\overline{v}_1), \overline{v}_1]$ (it is zero otherwise). Notice that $EU_1(v_1^0(\overline{v}_1)|\overline{v}_1) = v_1^0(\overline{v}_1)F_2(v_2^0(\overline{v}_1))$, since bidder 1 with type $v_1^0(\overline{v}_1)$ wins only if bidder 2 stays out, given his bid is zero.

If $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect Proposition 1 implies that $v_2^0(\overline{v}_1) = \underline{v}$, meaning that $EU_1(v_1^0(\overline{v}_1)|\overline{v}_1) = 0$, since a bid of zero by bidder 1 would have zero probability of winning. Furthermore since $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect, $v_1^0(\overline{v}_1) \leq v_1^0(\widehat{v}_1)$ and $k^{-1}(v|\widehat{v}_1)$ must exceed $k^{-1}(v|\overline{v}_1)$. The latter implies that $F_2(k^{-1}(v|\widehat{v}_1))$ is above $F_2(k^{-1}(v|\overline{v}_1))$. Consequently, in the counterpart to (24) for $\widehat{v}_1$ (after the truncation) the first term is no smaller, and the second term strictly larger than before the truncation. Thus, bidder 1's expected payoff increases.

If $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect, then $v_1^0(\overline{v}_1) \geq v_1^0(\widehat{v}_1)$ and $k^{-1}(v|\widehat{v}_1)$ must be below $k^{-1}(v|\overline{v}_1)$ for low types but above it for high types. An argument similar to the one above leads to the conclusion that bidder 1 is worse off if his type is smaller than the type where $k^{-1}(v|\widehat{v}_1) = k^{-1}(v|\overline{v}_1)$. To the right of this point, however, expected payoff is steeper after the truncation. Thus, there can be at most one point of crossing between expected payoff before and after the truncation. Whether the post-truncation payoff "catches up" therefore depends solely on whether $EU_1(\widehat{v}_1|\widehat{v}_1)$ is smaller or larger than $EU_1(\widehat{v}_1|\overline{v}_1)$. Example 2 proves that both are possible. ■

Proposition 7 *Expected revenue strictly decreases if $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect. If $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect, expected revenue may increase or decrease.*

Proof. Proposition 5 implies that the expected payment from bidder 1 decreases. If $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect, bidder 2 lowers his bid regardless of his type after the truncation, and expected payment from bidder 2 must therefore also decrease. Example 2 proves the second part of the proposition. ■

The above analysis assumes that bidder 1 becomes "weaker" exogenously. However, in many dynamic settings bidder 2 is able to infer that bidder 1 is weaker than he believed at the outset, simply by observing bidder 1 taking, or not taking, a specific action. We now turn to such situations.

4 Endogenous information update

In the following we analyze a larger game in which an information update of the form considered thus far arises endogenously. In particular, we assume bidder 1, after having learned his type, is made an offer to win the object outright at a price of B. If he declines the offer, an all-pay auction is held. B can be exogenously determined, or it can be determined by the seller.[13]

This simple game describes many situations, and B can be interpreted in several ways. For example, B could be interpreted as a preemptive bribe, in which case paying B allows bidder 1 to win the prize or object immediately without having to enter a contest or all-pay auction with bidder 2. Alternatively, B could be a performance threshold. For instance, an internal employee (bidder 1) is promoted without having to compete with an external candidate (bidder 2) if his recent performance is sufficiently good.

The game could also be modelled as one in which bidder 1 first submits a bid, b_1. Bidder 1 wins outright and the game stops if $b_1 \geq B$. Otherwise, bidder 2 is given the chance to submit a bid, b_2, without knowing b_1. The winner of the auction would then be the bidder with the highest bid. In both models, the crucial feature is that *if* bidder 2 is given the chance to compete, he will be able to infer that bidder 1 did not take a specific action.[14]

It is easily seen that the bribe is more likely to appeal to bidder 1 if his type is high. Assuming it is not prohibitively high, declining the bribe is therefore evidence that bidder 1's type is not too high after all. Thus, bidder 2 updates his beliefs, in the manner of a truncation.

If bidder 1 rejects the offer, the common maximal bid, $\bar{b}$, in the ensuing all-pay auction is a function of the revised beliefs. If $\bar{b} < B$ bidder 1 should never accept the pre-auction offer. The reason is that if he rejects the offer he could still win with probability one by bidding $\bar{b}$ in the auction, which is cheaper than accepting B. On the other hand, if $\bar{b} > B$ bidder 1 is irrationally

[13]See Kirkegaard and Overgaard (2007) for an analysis of first-price and second-price auctions with one or more pre-auction offers.

[14]The following is another closely related game: There is a cap on bids, and bidder 2 learns whether bidder 1 submitted this maximum bid before submitting a bid of his own. Bidder 2 can meet but not exceed the cap. If both bidders submits the maximum allowable bid, bidder 2 wins with a probability strictly less than one. If the cap is large or the winning probability is low, bidder 2 would be strictly better off giving up (bidding zero) rather than meeting the cap if he learns bidder 1 has met the cap. The analysis of this game would then be identical to the analysis of the other games just described. In contrast, the existing literature on caps in contests assume choices are made simultaneously.

rejecting the offer for a set of types that would bid close to $\bar{b}$ in the auction. These types would be better off accepting the lower pre-auction offer and winning with probability one.

Hence, when the offer is not prohibitively high equilibrium must necessarily satisfy $\bar{b} = B$. From the previous analysis we know that the equilibrium in the all-pay auction is unique and that the common maximal bid is strictly lower the lower the point of truncation is (Proposition 4). Hence, we can uniquely find bidder 1's critical type who is indifferent between accepting and rejecting the offer, and who defines the point of truncation, by solving

$$b_1(\widehat{v}_1|\widehat{v}_1) = B,$$

when $B < b_1(\overline{v}_1|\overline{v}_1) = \overline{b}_\varphi$. Thus, if $v > \widehat{v}_1$ bidder 1 will accept the offer. Otherwise he will reject the offer and compete in the all-pay auction instead. Notice that $\widehat{v}_1$ is strictly increasing in B when $B < \overline{b}_\varphi$. However, bidder 1 will always reject the offer if B is larger than $\overline{b}_\varphi$.

The primary aim of this section is to examine how bidders' payoffs are impacted by the pre-auction offer. Starting with bidder 1, it is obvious that he is strictly better off when his type is sufficiently high and the offer is not prohibitively high, $B < \overline{b}_\varphi$. If his type is $\overline{v}_1$ he would win with probability one with or without the offer, but he would pay less in the former case.

Moreover, in the case where bidder 2 responds to the information update by lowering his bid ($k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect), bidder 1 is strictly better off for all $v \in (v_1^0(\widehat{v}_1), \overline{v}_1)]$, i.e. for all types that submits strictly positive bids after the truncation.

On the other hand, Proposition 6 cautions that it is also possible that bidder 1 is made worse off if his type is low, because bidder 2 might react to the information update by becoming more aggressive for low types (Proposition 4; $k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect). In this case, bidder 2's aggressiveness deters more types of bidder 1 from entering the auction, and those types that do compete face a smaller probability of winning if low bids are submitted.

Indeed, it is possible that even a subset of the types *accepting* the offer are worse off being given the offer, as illustrated in Example 2 where type $\widehat{v}_1 = 2$ is potentially worse off. Although such a type wins with probability one, the increase in cost ($B > \widehat{b}_\varphi$) more than offsets this benefit.

Turning to bidder 2, Corollary 2 implies that bidder 2 is better off *contingent* on the all-pay auction being reached. However, in the present model bidder 2 suffers from the possibility that bidder 1 might preempt the auction by accepting the offer. As a consequence, bidder 2 might be better or worse off when a pre-auction offer is extended to bidder 1.

When bidder 1 increases his bid for all the types that reject the offer and instead participate in the auction ($k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ do not intersect), bidder 2 must be worse off. The reason is that he wins less often as bidder 1 either accepts the offer or submits higher bids.[15]

However, when bidder 1 bids less aggressively after the truncation for low types ($k(v|\widehat{v}_1)$ and $k(v|\overline{v}_1)$ intersect), bidder 2 must be better off for a set of types close to $v_2^0(\widehat{v}_1)$. In this case the pre-auction offer given to bidder 1 is advantageous to bidder 2 if his type is low for two reasons. First, if bidder 1 accepts the offer he must have a high type, implying that bidder 2 would most likely have lost the auction, and his bid, without the pre-auction offer. Second, if bidder 1 rejects the offer bidder 2 has cause to be more optimistic, knowing that any relatively low bid he submits has a higher probability of being successful.

However, if bidder 2 has a high type, and thus a high probability of winning an auction, his payoff might decline since bidder 1 might preempt the auction.

In conclusion, the introduction of a pre-auction offer either decreases bidder 2's expected payoff regardless of his type, or bidder 1's expected payoff declines if he happens to have a low type. Hence, the offer does not constitute an "interim" Pareto improvement.

Proposition 8 *Assume $B < \overline{b}_\varphi$. At least one bidder has a set of types for which expected payoff decreases with the introduction of a pre-auction offer.*

Nevertheless, the next example shows that the pre-auction offer might be an *ex ante* Pareto improvement. That is, it is possible that both bidders would prefer a pre-auction offer if given the choice before learning their types. The example shows that this can happen at the pre-auction offer which maximizes total expected expenditures. Hence, if there is a third party who receives a percentage of the total expenditure, he would also be better off ex ante.

Example 2 (continued): If bidder 1 is given a pre-auction offer but rejects it, bidder 2 learns that bidder 1's type is in an interval $[0, \widehat{v}_1]$. In the all-pay

[15] Hence, a formal proof of the fact that bidder 2 is worse off for all $v \in (\underline{v}, \overline{v}_2]$ follows the logic of the proof of Proposition 6. Bidder 2 with type v wins with probability $F_1(k(v|\widehat{v}_1))$ after the truncation, which is less than $F_1(k(v|\overline{v}_1))$. The result then follows from (24).

auction the common maximal bid is then $\frac{\widehat{v}_1}{1+\widehat{v}_1}$. Thus, it must be the case that $B = \frac{\widehat{v}_1}{1+\widehat{v}_1}$ to support this equilibrium. Since bidder 1 accepts B if his type is above $\widehat{v}_1$, the probability that the seller earns B is $1 - \frac{\widehat{v}_1}{\overline{v}_1}$. If bidder 1 rejects the offer, (21) and (22) can be used to calculate total revenue in the auction. The seller's overall expected revenue is

$$
\begin{aligned}
ER\left(\widehat{v}_1|\overline{v}_1\right) &= \left(1 - \frac{\widehat{v}_1}{\overline{v}_1}\right) B + \frac{\widehat{v}_1}{\overline{v}_1}\left(EP_1(\widehat{v}_1) + EP_2(\widehat{v}_1)\right) \\
&= \left(1 - \frac{\widehat{v}_1}{\overline{v}_1}\right) \frac{\widehat{v}_1}{1+\widehat{v}_1} + \frac{\widehat{v}_1}{\overline{v}_1}\left(\frac{\widehat{v}_1^2}{3\widehat{v}_1 + 2\widehat{v}_1^2 + 1} + \frac{\widehat{v}_1}{3\widehat{v}_1 + \widehat{v}_1^2 + 2}\right) \\
&= \frac{1}{\overline{v}_1}\frac{\widehat{v}_1}{1+\widehat{v}_1}\left(\overline{v}_1 - \widehat{v}_1 + \frac{\widehat{v}_1^2}{1+2\widehat{v}_1} + \frac{\widehat{v}_1}{2+\widehat{v}_1}\right).
\end{aligned}
$$

Figure 5 illustrates $ER\left(\widehat{v}_1|\overline{v}_1\right)$ for various levels of $\overline{v}_1$. It is seen that a pre-auction offer is profitable to the seller if it *reduces* the asymmetry between the bidders in the all-pay auction, although the asymmetry should not be reduced too much.[16]

In particular, if $\overline{v}_1 > \overline{v}_2 = 1$, a fairly high pre-auction offer is beneficial to the seller. However, if $\overline{v}_1 < 1$, the seller should not give bidder 1 an offer. Instead, the seller would benefit from giving bidder 2 an offer, thereby reducing the asymmetry in the auction, should it come about. The only situation in which the seller is unable to profit from a pre-auction offer to some bidder is if the two bidders are homogeneous to begin with, $\overline{v}_1 = \overline{v}_2$.

Since the revenue-maximizing pre-auction offer serves to reduce the asymmetry among bidders, bidders use strategies that are more similar in the auction. Moreover, if the offer is accepted it must be because the bidder has a very high type, implying that it is indeed efficient for him to win the object. Therefore, social surplus, as measured by the expected valuation of the winner, will increase.[17,18]

[16] A special feature of this example is that one bidder's distribution is a truncated version of the distribution of the other bidder (both are uniform). In this case it is meaningful to argue that the update in beliefs can reduce the asymmetry.

[17] The revenue-maximizing pre-auction offer reduces the asymmetry between bidders, but does not "reverse" it (bidder 1 remains stronger, $\widehat{v}_1 > 1$, in the all-pay auction). It can be shown that social surplus is single-peaked in $\widehat{v}_1$, attaining its maximum at $\widehat{v}_1 = \overline{v}_2 = 1$, where the bidders are symmetric in the all-pay auction. Thus, the revenue-maximizing offer must increase social surplus.

[18] The details of the example are omitted but available upon request.

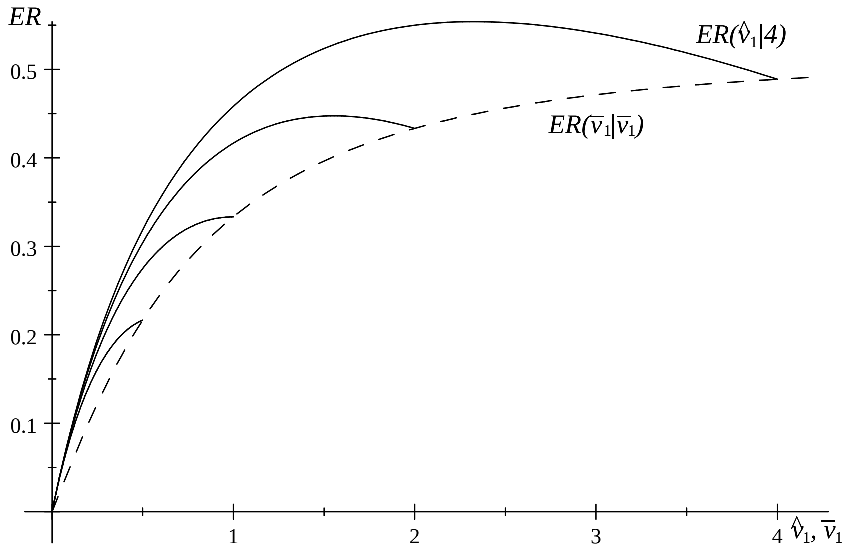

Figure 5: $ER(\widehat{v}_1|\overline{v}_1)$ for $\overline{v}_1 = 0.5, 1, 2, 4$, and total expected revenue without the pre-auction offer, $ER(\overline{v}_1|\overline{v}_1)$.

However, each bidder has types for which he is better off in the game with a pre-auction offer and types for which the opposite is the case. It might therefore be of some interest to consider the *ex ante* payoff of the bidders. Ex ante payoff to the bidders as a function of $\widehat{v}_1$ and $\overline{v}_1$ are,

$$\begin{aligned}
EU_1(\widehat{v}_1|\overline{v}_1) &= \int_0^{\widehat{v}_1} \left(vk^{-1}(v|\widehat{v}_1) - b_1(v|\widehat{v}_1)\right)\frac{1}{\overline{v}_1}dv + \int_{\widehat{v}_1}^{\overline{v}_1}(v-B)\frac{1}{\overline{v}_1}dv \\
EU_2(\widehat{v}_1|\overline{v}_1) &= \frac{\widehat{v}_1}{\overline{v}_1}\int_0^{\overline{v}_2}\left(v\frac{k(v|\widehat{v}_1)}{\widehat{v}_1} - b_2(v|\widehat{v}_1)\right)dv.
\end{aligned}$$

Bidder 1 wins the object at a price of B if his valuation is above $\widehat{v}_1$. Otherwise, he rejects the offer and bids $b_1(v|\widehat{v}_1)$ in the auction, which means he will outbid bidder 2 if bidder 2's type is below $k^{-1}(v|\widehat{v}_1)$. Bidder 2, on the other hand, will get a chance to win the object only if bidder 1 rejects the offer. $EU_1(\widehat{v}_1|\overline{v}_1)$ and $EU_2(\widehat{v}_1|\overline{v}_1)$ can be reduced to

$$\begin{aligned}
EU_1(\widehat{v}_1|\overline{v}_1) &= \frac{\widehat{v}_1^4}{\overline{v}_1(\widehat{v}_1+1)(2\widehat{v}_1+1)} + \frac{1}{\overline{v}_1}\left(\frac{1}{2}\left(\overline{v}_1^2 - \widehat{v}_1^2\right) - \frac{\widehat{v}_1}{\widehat{v}_1+1}\left(\overline{v}_1 - \widehat{v}_1\right)\right) \\
EU_2(\widehat{v}_1|\overline{v}_1) &= \frac{\widehat{v}_1}{\overline{v}_1(1+\widehat{v}_1)(\widehat{v}_1+2)}.
\end{aligned}$$

Figure 6 summarizes the relevant information. It contains an "indifference curve" for each bidder, on which the bidder is indifferent between the game with no pre-auction offer and the game with such an offer, $EU_i(\widehat{v}_1|\overline{v}_1) = EU_i(\overline{v}_1|\overline{v}_1)$, $i = 1, 2$. For any given $\overline{v}_1$, bidder 1 is better off, ex ante, below the indifference curve, as he would prefer a lower pre-auction offer. The opposite holds for bidder 2, who ex ante dislikes low pre-auction offers since they are more likely to be accepted by the competition. Notice there is a region where bidder 1 is better off and bidder 2 worse off, another region with the reverse property, and finally a region in which *both* bidders are better off. In all cases, at least one bidder is better off ex ante with the introduction of a pre-auction offer.

The revenue-maximizing pre-auction offer to bidder 1 might be of particular interest, and hence it is also indicated in Figure 6.[19] If $\overline{v}_1 \leq 1$ the seller is better off not extending an offer to bidder 1, as noticed earlier. However, if bidder 1 is stronger than bidder 2, $\overline{v}_1 > 1$, the seller benefits from giving bidder 1 a pre-auction offer. If the asymmetry is small or very large, one bidder benefits, ex ante, while the other is hurt. However, if the asymmetry is in an intermediate range ($\overline{v}_1$ between 1.52 and 3.66) then not only is the seller better off, so are both bidders. Hence, a pre-auction offer might constitute an ex-ante Pareto improvement. Recall that even when one bidder is hurt by the optimal pre-auction offer, social surplus nevertheless increases.

Incidentally, the last property is in contrast to a central result in Clark and Riis (2000). Using the same distributional assumptions, they examine a different way of discriminating between bidders. In particular, they show the seller benefits from allowing bidder 2 to win even if his bid is a certain percentage below bidder 1's bid (when $\overline{v}_1 > 1$). However, since bidder 1 is more likely to value the object higher, it would in fact be more efficient if bidder 1 was given preferential treatment rather than bidder 2. Hence, in Clark and Riis (2000) the seller's optimal design leads to a decrease in social surplus. $\square$

[19] More precisely, the optimal cut-off ($\widehat{v}_1$) is indicated. The optimal pre-auction offer is then $\frac{\widehat{v}_1}{1+\widehat{v}_1}$.

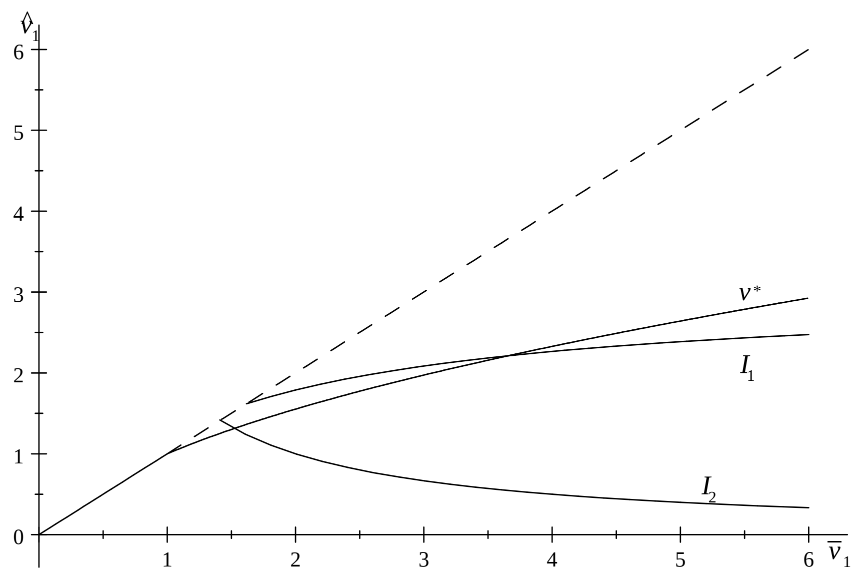

Figure 6: "Ex ante indifference curves", I_1 and I_2, in $(\overline{v}_1, \widehat{v}_1)$ space for bidder 1 and 2, respectively, and the revenue-maximizing truncation, v^*.

We conclude with a formal statement of the implication of the example.

Proposition 9 *Assume $B < \overline{b}_{\varphi}$. The pre-auction offer may leave the seller and both bidders better off ex ante.*

5 Conclusion

A contestant's perception of the competition is all-important in determining his behavior in the contest. In this paper we analyzed how behavior changes when beliefs change. A contestant is more likely to participate in the contest the weaker the competition is, but it is not necessarily the case that a contestant who is considered to have become weaker will be less inclined to compete. Indeed, it is possible that the latter contestant will benefit when his competitor takes him to be weaker. Moreover, total expenditure by the contestants may increase or decrease when one becomes weaker.

In certain dynamic environments contestants update their beliefs as time passes. We studied a particular class of games in which a revision of beliefs of the form considered before to be exogenous may take place endogenously. Examples include dynamic contests with bribes.

In these games, at least one of the contestants has a type who is made worse off. In particular, it is possible that the first to move is disadvantaged if his valuation is low. Nevertheless, we demonstrated that the existence of an option to bribe may be welfare improving. In fact, it is possible that all parties involved – the contestants and the recipient of the effort – are better off ex ante.

References

[1] Amann, E. and Leininger, W., 1996, Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case, *Games and Economic Behavior*, 14, 1-18.

[2] Burguet, R. and Perry, M.K., 2007, Bribery and Favoritism by Auctioneers in Sealed-Bid Auctions, *The B.E. Journal of Theoretical Economics*, 7 (1): Article 23 (Contributions).

[3] Chan, W., 1996, External Recruitment versus Internal Promotion, *Journal of Labor Economics*, 14 (4): 555-570.

[4] Che, Y-K and Gale, I.L., 1998, Caps on Political Lobbying, *American Economic Review*, 88 (3): 643-651.

[5] Che, Y-K and Gale, I.L., 2006, Caps on Political Lobbying: Reply, *American Economic Review*, 96 (4): 1355-1360.

[6] Chen, K.-P., 2005, External Recruitment as an Incentive Device, *Journal of Labor Economics*, 23 (2): 259-277.

[7] Clark, D.J. and Riis, C., 2000, Allocation efficiency in a competitive bribery game, *Journal of Economic Behavior & Organization*, 42: 109-124.

[8] Gavious, A., Moldovanu, B., and Sela, A., 2002, Bid costs and endogenous bid caps, *RAND Journal of Economics*, 33 (4): 709-722.

[9] Hopkins, E. and Kornienko, T., 2007, Cross and Double Cross: Comparative Statics in First Price and All Pay Auctions, *The B.E. Journal of Theoretical Economics*, 7 (1): Article 19 (Topics).

[10] Kaplan, T.R. and Wettstein, D., 2006, Caps on Political Lobbying: Comment, *American Economic Review*, 96 (4): 1351-1354.

[11] Kirkegaard, R. and Overgaard, P.B., 2007, Pre-Auction Offers in Asymmetric First-Price and Second-Price Auctions, mimeo.

[12] Lebrun, B., 1998, Comparative Statics in First Price Auctions, *Games and Economic Behavior*, 25: 97-110.

[13] Lien, D., 1990, Corruption and Allocation Efficiency, *Journal of Development Economics*, 33: 153-164.

[14] Parreiras, S.O. and Rubinchik, A., 2006, Contests with Many Heterogeneous Agents, mimeo.

[15] Sahuguet, N., 2006, Caps in asymmetric all-pay auctions with incomplete information, *Economics Bulletin*, 3 (9): 1-8.