A Simple Approach to Analyzing Asymmetric First Price Auctions*

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Abstract

In this paper, we propose a new approach to analyzing asymmetric first price auctions. Specifically, we examine winning probabilities, exploiting the connection between winning probabilities and payoffs known from mechanism design. This circumvents the need to look directly at bidding strategies, which are generally complex. We derive new results, and more easily prove almost all existing results. The approach also sheds light on hitherto unexamined types of asymmetry. Specifically, we consider auctions with predictable and unpredictable buyers. Moreover, the method also applies to asymmetric all-pay auctions, where all buyers pay their own bid, and about which little is currently known.

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1 Introduction

Though the rules of a first price auction are simple, the same cannot necessarily be said for equilibrium behavior, especially when buyers are asymmetric. In fact, the existence of an equilibrium was not established until recently.[1] Since then, a few properties of bidding behavior have been uncovered, including, for example, the tendency for weak buyers to bid more aggressively than strong buyers.[2]

When analyzing first price auctions it is natural to start by examining bidding strategies directly, as is done in the existing literature. However, this approach is made difficult by the fact that equilibrium bidding strategies derive from a system of differential equations (arising from the first order conditions), which generally escapes explicit solution. In this paper, we will sidestep these difficulties by exploring another method of analyzing asymmetric first price auctions.

More specifically, the point of departure in this paper will be the probability that a given buyer with a given type, or valuation, will win the auction in equilibrium. Compared to bidding strategies, it is much simpler to analyze winning probabilities, and it involves no manipulation of systems of differential equations. In addition, the winning probabilities are arguably just as interesting as the bidding strategies, yet the standard approach reveals little about these.

Given the winning probabilities, we will subsequently be able to infer a number of properties of the bidding strategies. The winning probabilities will also inform us of buyers' preferences for different auction formats, and the type of competition they face. In fact, most existing results can be established more easily using the method proposed here. Hence, the paper

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[1] See Lebrun (1999) and Maskin and Riley (2000b). Earlier papers contained examples in which explicit derivations of bidding strategies are possible. See Vickrey (1961), Greismer et. al. (1967), and Plum (1992). Marshall et. al. (1994) use numerical methods to estimate bidding strategies. Riley and Samuelson (1981) derived bidding strategies in the symmetric case.

[2] See Lebrun (1999), Maskin and Riley (2000a), and, for a related point, Fibich et. al. (2002). Maskin and Riley (2000a) also show that there is no unambiguous ranking in terms of revenue of the first and second price auctions. Lebrun (1998) examines the consequences on revenue and bidding when one buyer becomes stronger, while Cantillon (2004) compares revenue for different degrees of asymmetry among buyers. Fibich and Gavious (2003) use perturbation methods to study asymmetries. See Krishna (2002) for an introduction to asymmetric auctions.

also serves to synthesize and unify the literature on asymmetric first price auctions.

At the heart of the approach taken in this paper is the property that knowledge of the winning probabilities is sufficient to determine expected payoff to buyers. This follows from the incentive compatibility constraint, and is a well known result in mechanism design (see Section 2). Then, since payoff is a function of bid and winning probability, it is possible to deduce what the bid is, given the winning probability is known.

In order to examine winning probabilities, we utilize the property that the highest bid is the same for all buyers, meaning that buyers with the highest possible valuation are equally well off, regardless of their identities. In Section 2, this allows us to establish that no one buyer wins consistently more often than another. That is, if winning probabilities are compared across buyers, they will not be uniformly higher, as a function of valuation, for one buyer.

In Section 3, we consider auction environments where one buyer is stronger than another, in the sense of first order stochastic dominance. That is, the strong buyer has a higher probability of having a high valuation. Almost all existing literature on asymmetric first price auctions studies this particular form of asymmetry.

The central result in this case is that the strong buyer's winning probability is a mean preserving spread over the winning probability of the weak buyer. This result is equivalent to the result that the strong buyer is consistently better off than the weak buyer, i.e. the payoff of the former is higher regardless of valuation.

Furthermore, it is possible to bound the winning probabilities. In concert with the previous result, this allows us to provide new proofs of most existing results, and, in some cases, to offer generalizations.

Having established the simplicity of the approach in a well known environment, we briefly explore two other directions in which the approach can be applied to yield new insights. First, in Section 4, we consider another compelling type of asymmetry among buyers in the first price auction. Specifically, we compare buyers whose valuations are either relatively predictable or unpredictable, and show that the division of surplus among buyers in the first price auction is dramatically different from the division in second price auctions. Furthermore, the buyer whose valuation is relatively predictable also submits a bid that is relatively predictable.

Then, in Section 5, we illustrate that the approach taken in this paper may also be useful in other types of auctions than the first price auction. Specifically, the approach is also applicable to all-pay auctions in which buyers pay their bid regardless of whether they win or lose the auction. Studying this type of auction also leads to insights into patent competitions, lobbying activity, and similar environments that can be likened to tournaments. However, with the notable exception of Amann and Leininger (1996), the all-pay auction with asymmetric buyers and incomplete information has not been subject to much study. In this paper, we contribute to the theory of all-pay auctions by showing that all-pay auctions with weak and strong buyers share many features with the first price auction, although there are some important exceptions. These results appear to be previously unknown.

Section 6 offers some concluding remarks.

2 Model and preliminaries

We consider a first price auction with n risk neutral and potentially asymmetric buyers with independent private values. Buyer i draws his valuation, v_i, from the distribution function F_i, on $v_i \in [0, \overline{v}]$, $i = 1, ..., n$. F_i is assumed to have no mass points, and to be strictly increasing and continuously differentiable, with $f_i > 0$ denoting the density. We let $b_i(v)$ denote buyer i's equilibrium bidding strategy, and we let $q_i(v)$ denote the equilibrium probability that buyer i with valuation v will win the auction when following his equilibrium strategy.[3]

In this environment, Lebrun (1999) and Maskin and Riley (2000b) have shown that an equilibrium exists, and that bidding strategies are continuous and strictly increasing in valuations.[4] Moreover, bidding strategies are differentiable on $(0, \overline{v}]$. Finally, Lebrun (1999) has shown that $b_i(0) = 0$, implying that $q_i(0) = 0$, for all i, $i = 1, 2, ..., n$. Since $b_i(v)$ is increasing and differentiable on $(0, \overline{v}]$, so is $q_i(v)$. We will take these relatively intuitive properties as given, and examine other properties of the first price auction.

There are two ways to derive expected payoff to a given buyer, both of

[3]Hence, $q_i(v)$ depends on the equilibrium bid of buyer i and the equilibrium bidding strategies of buyer i's rivals.

[4]Lebrun (1999) also addresses uniqueness, as do Maskin and Riley (2003). If there is a reserve price, the equilibrium is unique. The same holds in a second price auction, as shown by Blume and Heidhues (2004). Lebrun (2005) establishes uniqueness under the condition that F_i is log-concave at $v = 0$ for all i, $i = 1, 2..., n$.

which are useful. It is straightforward to see that, in equilibrium, buyer i with valuation v has expected payoff of

$$EU_i(v) = (v - b_i(v))\, q_i(v), \tag{1}$$

since his payoff is $v - b_i(v)$ if he wins, which he does with probability $q_i(v)$.

However, inspired by Myerson's (1981) mechanism design method, it is useful to approach expected payoff from a different angle. If buyer i with valuation v chooses to bid $b_i(z)$, his payoff will be $(v - b_i(z))q_i(z)$. Since he submits the bid which maximizes his payoff, we can write

$$EU_i(v) = \max_z (v - b_i(z))q_i(z).$$

As buyer i with valuation v bids $b_i(v)$ in equilibrium, the problem is maximized at $z = v$.[5] This holds for arbitrary v, and we can now use the Envelope Theorem (while considering v to be a parameter), to conclude that

$$EU_i'(v) = q_i(v). \tag{2}$$

It follows that expected payoff can also be expressed as

$$EU_i(v) = \int_0^v q_i(x)dx.^6 \tag{3}$$

We will use (2) and (3) extensively in the following.

By comparing (1) and (3), the link between winning probabilities and bids becomes apparent,

$$b_i(v) = v - \int_0^v \frac{q_i(x)}{q_i(v)} dx. \tag{4}$$

Clearly, buyer i with valuation v bids below his valuation, a phenomenon usually referred to as bid shading.

[5] That is, in equilibrium the buyer submits the bid he is "supposed" to submit given his valuation. Deviating to another bid, or behaving as if his valuation was different, is not profitable.

[6] (2) and thus (3) can also be established by methods that do not require differentiability of $q_i(v)$. Notice that the constant of integration is zero, as a buyer with valuation 0 has payoff 0.

Importantly, Lebrun (1999) has also shown that the bid submitted by a buyer with valuation $\overline{v}$ is the same for all buyers.[7,8] We will let $\overline{b}$ denote this bid. Since the highest bid is the same among all buyers, it follows that $q_i(\overline{v}) = 1$ for all i, and that buyers with valuation $\overline{v}$ are equally well off, or, for $j \neq i$,

$$\int_0^{\overline{v}} q_i(x)dx = EU_i(\overline{v}) = \overline{v} - \overline{b} = EU_j(\overline{v}) = \int_0^{\overline{v}} q_j(x)dx. \tag{5}$$

It follows from (5) that it is not possible for buyer i to win consistently more often than buyer j, or $q_i(v) > q_j(v)$ for all $v \in (0, \overline{v})$. If this was the case, the term on the far left in (5) would strictly exceed that on the far right, giving rise to a contradiction.

Proposition 1 *In a first price auction, buyer i will never win consistently more often than buyer j, $i \neq j$, regardless of F_i and F_j. That is, it is not possible for $q_i(v) > q_j(v)$ for all $v \in (0, \overline{v})$.*

Proposition 1 implies that q_i and q_j either coincides, as is the case with symmetric buyers, or cross at least once.[9]

Thus, even a very strong buyer will not win consistently more often than a weaker buyer, presumably because the former, realizing his strength, shades his bid more below his valuation. On the other hand, while the weaker buyer may be more aggressive, this is not sufficient to lead him to win consistently more often, in equilibrium. Section 3 examines the interaction between strong and weak buyers in more detail.

[7] Since this plays an important role in the following, we prove it here. By contradiction, assume that $b_1(\overline{v}) \geq b_2(\overline{v}) \geq ... \geq b_n(\overline{v})$, with at least one strict inequality. Then, since it is supposedly optimal for buyer n to bid $b_n(\overline{v})$ when his valuation is $\overline{v}$, we conclude his equilibrium payoff is no smaller than $\overline{v} - b_1(\overline{v})$, the payoff he would get from imitating buyer 1. This coincides with the equilibrium payoff to buyer 1 with valuation $\overline{v}$, and so in turn can be no smaller than what buyer 1 would get from bidding $b_n(\overline{v})$. However, if buyer 1 bids $b_n(\overline{v})$, he earns strictly higher payoff than if buyer n were to bid $b_n(\overline{v})$, i.e. his equilibrium payoff is higher than buyer n's equilibrium payoff. The reason is that a bid of $b_n(\overline{v})$ is sure to beat buyer n, but not buyer 1. Consequently, we have a contradiction.

[8] When the distribution functions have different support, the highest bid is also the same in auctions with *two* buyers, see Maskin and Riley (2000a). Most of the following results remain valid in this set-up as well.

[9] Lebrun (1999) shows that symmetric buyers must use the same strategy in equilibrium.

3 Strong and weak buyers

In the following, we will compare weak and strong buyers. While there may still be n buyers, we will focus on two of these, buyer s and buyer w. We assume that buyer s is strong relative to buyer w, who we will term weak. Formally, we assume that F_s *first order stochastically dominates* F_w, or $F_s(v) < F_w(v)$, for all $v \in (0, \overline{v})$. That is, buyer s is more likely to have a high valuation.

For future use, define $H(b)$ as the distribution function of the highest bid among the $n-2$ rivals to buyer s and buyer w. Furthermore, define $q_s^w(v)$ as the probability that buyer s with valuation v outbids buyer w, and define $q_w^s(v)$ analogously. Then, $q_s(v) = H(b_s(v))q_s^w(v)$, i.e. buyer s with valuation v wins if he outbids buyer w, as well as everybody else.

To compare payoffs to the two buyers we define

$$R(v) \equiv \frac{EU_s(v)}{EU_w(v)} = \frac{(v - b_s(v))\, H(b_s(v))q_s^w(v)}{(v - b_w(v))\, H(b_w(v))q_w^s(v)}, \qquad v \in (0, \overline{v}] \tag{6}$$

as the ratio of payoffs, with

$$R'(v) = \frac{EU_s'(v)EU_w(v) - EU_w'(v)EU_s(v)}{EU_w(v)^2} = \frac{q_s(v)q_w(v)\,[b_s(v) - b_w(v)]}{EU_w(v)^2}, \tag{7}$$

where the last equality follows from (1) and (2).

There is an important relationship between $R(v)$ and the bidding strategies. If $b_s(v) = b_w(v)$, then $q_s^w(v) = F_w(v)$ since buyer s with valuation v outbids buyer w if buyer w has valuation below v, which occurs with probability $F_w(v)$. Likewise, $q_w^s(v) = F_s(v)$ and we conclude that $R(v) = F_w(v)/F_s(v)$.

On the other hand, if $b_s(v) > b_w(v)$ then (i) $q_w^s(v) < F_s(v)$ since buyer w with valuation v would lose to buyer s with valuation v, and (ii) buyer s is better off following his equilibrium strategy than bidding $b_w(v)$, or $EU_s(v) \geq (v - b_w(v))\, H(b_w(v))F_w(v)$, where the last term on the right comes from the fact that buyer s outbids buyer w with valuation below v by bidding $b_w(v)$. The implication is that $R(v) > F_w(v)/F_s(v)$. Finally, the inequality is reversed if $b_s(v) < b_w(v)$ since in this case $q_s^w(v) < F_w(v)$ and $EU_w(v) \geq (v - b_s(v))\, H(b_s(v))F_s(v)$.

In summary, $R(v) \geq F_w(v)/F_s(v)$ *if, and only if,* $b_s(v) \geq b_w(v)$. Further, if $b_s(v)$ is larger (smaller) than $b_w(v)$ then $R(v)$ is increasing (decreasing), by (7). Figure 1 summarizes these findings. It is useful to remember that $R(\overline{v}) = 1 = F_w(\overline{v})/F_s(\overline{v})$.

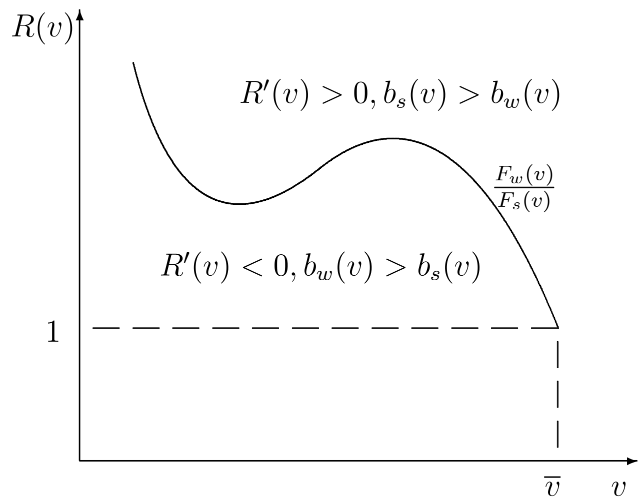

Figure 1: Relationship between $R(v)$ and bids.

We start with an analysis of payoffs and winning probabilities, and thereafter turn to the implications for bidding strategies and preferences.

3.1 Payoffs and winning probabilities

To compare winning probabilities, we will begin by showing that buyer s is consistently better off than buyer w, or $EU_s(v) > EU_w(v)$ for all $v \in (0, \overline{v})$. In other words, $R(v) > 1$ for all $v \in (0, \overline{v})$. By contradiction, assume that there is some value, x, $x \in (0, \overline{v})$, for which $R(x) \leq 1 < F_w(x)/F_s(x)$. Then, $b_w(x) > b_s(x)$ and $R'(x) < 0$. Hence, $R(\cdot)$ remains strictly below 1 and $F_w(\cdot)/F_s(\cdot)$ to the right of x, contradicting the fact that $R(\overline{v}) = 1$.[10]

Theorem 1 states the result formally. Notice that q_i can be likened to a

[10] An alternative proof examines the difference between payoff to the strong and weak buyers, $D(v) \equiv EU_s(v) - EU_w(v)$. Notice that $D(v)$ is continuously differentiable everywhere, since $D'(v) = q_s(v) - q_w(v)$. At the endpoints, the two buyers are equally well off, $D(0) = D(\overline{v}) = 0$. However, at any interior extremum, x, it must be the case that $q_s(x) = q_w(x)$, which leads us to conclude that $b_w(x) > b_s(x)$. Conversely, assume for instance that $b_s(x) = b_w(x)$. Then, buyer s with valuation x will outbid buyer w if buyer w has a valuation below x, which occurs with probability $F_w(x)$. Similarly, buyer w with valuation x outbids buyer s with probability $F_s(x)$. Hence, buyer w would win *less often* than buyer s, since $F_s(x) < F_w(x)$. We conclude that $q_s(x) = q_w(x)$ and $b_w(x) > b_s(x)$, and so (1) implies that $EU_s(x) > EU_w(x)$, or $D(x) > 0$. Consequently, $D(v)$ is minimized at the corners, meaning that $D(v) > 0$ for all $v \in (0, \overline{v})$.

distribution function, as it ranges from 0 to 1, and is increasing.

Theorem 1 *Buyer s is consistently better off than buyer w, or $EU_s(v) > EU_w(v)$ for all $v \in (0, \overline{v})$. In other words, q_s is a mean preserving spread over q_w. That is,*

$$\int_0^v q_s(x)dx > \int_0^v q_w(x)dx \tag{8}$$

for all $v \in (0, \overline{v})$, with equality at the endpoints.[11]

The second part of the Theorem follows from (3). Given Theorem 1, Figure 2 sketches the relationship between the winning probabilities of the strong and weak buyers.[12] Hence, we can conclude that although buyer s is consistently better off than buyer w, he does not win consistently more often. Intuitively, if the strong buyer has a high valuation, it is unlikely that the weak rival has a high valuation too, and so it is profitable for the strong buyer to shade his bid more. Therefore, if the weak buyer should have a high valuation, he is likely to win. On the other hand, if the strong buyer has a low valuation there is less of an incentive to shade his bid, and he wins more often than the weak buyer, contingent on a low valuation.

Theorem 1 allows us to bound the winning probabilities. Specifically, buyer s is better off than buyer w. In addition, buyer w with valuation v prefers bidding $b_w(v)$ rather than $b_s(v)$, since the former is the equilibrium strategy. Formally, we have the following chain

$$EU_s(v) = (v - b_s(v))\, H(b_s(v)) q_s^w(v) > EU_w(v) \geq (v - b_s(v)) H(b_s(v)) F_s(v),$$

where the last inequality follows from the fact that if buyer w bids $b_s(v)$, he will outbid buyer s only if buyer s has a valuation below v.

Importantly, we can conclude that $q_s^w(v) > F_s(v)$ for all $v \in (0, \overline{v})$. This, in turn, can be used to bound $q_w^s(v)$. These bounds on winning probabilities will be extremely useful in the following.

Lemma 1 *For all $v \in (0, \overline{v})$, $q_s^w(v) > F_s(v)$ and $q_w^s(v) < F_w(v)$.*

[11] q_w second order stochastically dominates q_s *and* the mean is the same. The latter can be seen by the fact that (8) holds with equality at $v = \overline{v}$ (integrate by parts to see that $EU_s(\overline{v})$ equals $\overline{v}$ less the expected value of a draw from q_s). See e.g. Krishna (2002), Appendix B.

[12] Formally, we have not shown that the two curves cross only once. However, they must cross an odd number of times, or infinitely often.

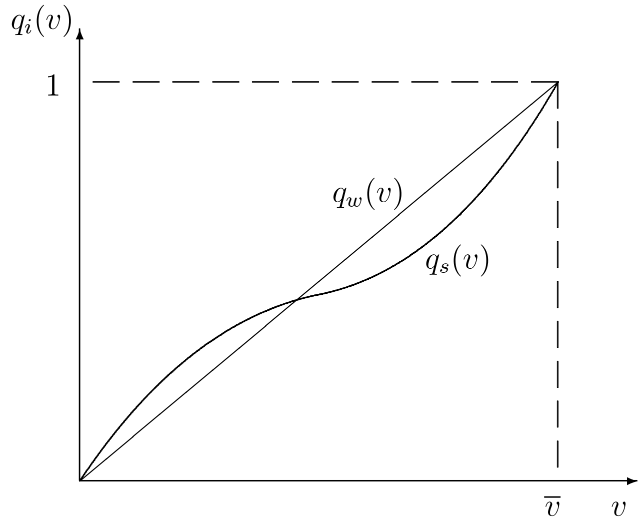

Figure 2: Sketch of winning probabilities of a strong and a weak buyer.

Proof. The first part was proven above. For any v, it implies that buyer s with valuation v submits the same bid as buyer w with some valuation z, where $q_s^w(v) = F_w(z) > F_s(v)$. From buyer w's point of view, this implies that if he has valuation z, he will bid the same as buyer s with valuation v, and thus outbid buyer s with probability $q_w^s(z) = F_s(v) < F_w(z)$. ■

In the next sub-sections, we will look at the implications for bidding strategies and preferences. Incidentally, this also leads to the insight that the strong buyer is more likely to win than the weak buyer, ex ante. Most of the results in the following sub-sections are known, but the method of proof is different, and arguably simpler.

3.2 Strategies

To compare bidding strategies we start by interpreting the fact that $q_s^w(v) > F_s(v)$ (Lemma 1). Buyer s with valuation v outbids buyer w if buyer w bids below $b_s(v)$. Since $b_s(v)$ is strictly increasing, $F_s(v)$ coincides with the probability that buyer s bids below $b_s(v)$. Therefore, the inequality $q_s^w(v) > F_s(v)$ can be written as

$$\Pr(w \text{ bids below } b_s(v)) = q_s^w(v) > F_s(v) = \Pr(s \text{ bids below } b_s(v)). \quad (9)$$

In other words, for any bid b, $b \in (0, \overline{b})$, the probability that w bids below b is higher than the probability that s does so.

Proposition 2 *The distribution of bids submitted by the strong buyer first order stochastically dominates the distribution of bids submitted by the weak buyer.*

Hence, by looking at payoffs and winning probabilities, we have provided a new proof of an important existing result. Proposition 2 is usually proven by examining the system of differential equations characterizing the bidding strategies.[13,14]

Although the weak buyer may bid more aggressively than the strong buyer, a point we will return to momentarily, this is not enough to outweigh the fact that the strong buyer is more likely to have a high valuation, and therefore a high bid. However, as we have seen, this does not mean that the strong buyer is more likely to win contingent on a given valuation.

In fact, the possibility that the weak buyer may be more likely to win, by bidding more aggressively, is sometimes highlighted. However, the first price auction format is not sufficient to completely "level the playing field", in the sense that the strong buyer is still more likely, ex ante, than the weak buyer to win the auction.

To see this, start with the following thought experiment. Rather than drawing a valuation from the distribution function (and subsequently calculating the bid), we can think of a buyer as drawing a bid from the distribution of bids (and later inferring the valuation, if necessary). This, of course, leads to the insight that the strong buyer is more likely to "draw", or submit, a high bid, and it follows that the strong buyer is more likely than the weak buyer to win the auction ex ante.

Corollary 1 *The ex ante probability that the auction is won by the strong buyer is strictly higher than the ex ante probability that the auction is won by the weak buyer.*

On a number of occasions, we have mentioned the possibility that the weak buyer bids more aggressively than the strong buyer. We are now ready to substantiate this claim. First, notice that at any point where $q_w(v) >$

[13] See Lebrun (1999), or Maskin and Riley (2000a).

[14] To prove this result, we used the fact that buyer s is consistently better off than buyer w (Theorem 1). This fact has, to our knowledge, not been explicitly stated before, although it actually follows from the known result in Proposition 2. Since buyer s faces a more benign distribution of rival bids, he must be better off than buyer w (and Theorem 1 follows).

$q_s(v)$, the weak buyer must necessarily be bidding more aggressively than the strong buyer, or he would not win with a higher probability. Hence, the weak buyer's bid must exceed the bid of the strong buyer for at least some valuations.

However, in order to globally compare bids, it is necessary to impose further assumptions. Following Lebrun (1999) and Maskin and Riley (2000a), assume that F_s dominates F_w in terms of the reverse hazard rate, or

$$\frac{f_s(v)}{F_s(v)} > \frac{f_w(v)}{F_w(v)} \text{ for all } v \in (0, \overline{v}]. \tag{10}$$

(10) is equivalent to the statement that $F_w(v)/F_s(v)$ is strictly decreasing. The condition is stronger than first order stochastic dominance, and has been shown, by the aforementioned authors, to imply that $b_w(v) > b_s(v)$ for all $v \in (0, \overline{v})$.[15] By examining $R(v)$, an alternative proof can be provided.

Proposition 3 *If (10) holds, then $b_w(v) > b_s(v)$ for all $v \in (0, \overline{v})$.*

Proof. In contradiction to the Proposition, assume that $b_s(v) \geq b_w(v)$ for some $v \in (0, \overline{v})$. Then, $R(v) \geq F_w(v)/F_s(v)$ and $R'(v) \geq 0$. However, $F_w(v)/F_s(v)$ is strictly decreasing, by (10). Consequently, $R(\cdot)$ is strictly above $F_w(\cdot)/F_s(\cdot)$ immediately to the right of v, implying that $b_s(\cdot) > b_w(\cdot)$ immediately to the right of v. In other words, $R(\cdot)$ and $F_w(\cdot)/F_s(\cdot)$ diverge (the former being larger and increasing) meaning that $b_s > b_w$ from v onwards. However, this contradicts the fact that $R(\overline{v}) = F_w(\overline{v})/F_s(\overline{v}) = 1$, or $b_s(\overline{v}) = b_w(\overline{v})$. ■

Fibich et. al. (2002) compare buyers that are strong and weak near the top, assuming only that $f_s(\overline{v}) > f_w(\overline{v})$.[16] By examining the system of differential equations at $\overline{v}$, they prove that the weak buyer bids more aggressively than the strong buyer near the top, i.e. for sufficiently high v.[17] The method of proof in Proposition 3 can be used directly to provide an alternative proof of this result, since the assumption implies that $F_w(v)/F_s(v)$ is strictly decreasing near the top.[18]

[15] See also Krishna (2002).

[16] Hence, they assume only that F_s is strictly smaller than F_w near $\overline{v}$, but not globally.

[17] Fibich et. al. (2002) also examine bidding close to the lower endpoint, and find that bidding strategies are approximately symmetric. This is consistent with our finding that the strong buyer wins more often for valuations close to the lower endpoint.

[18] The method of proof of Proposition 3 can also be applied to prove that symmetric buyers must use symmetric strategies.

In Theorem 1 we compared expected payoff to the buyers given first order stochastic dominance. Proposition 3 can be used in similar fashion given reverse hazard rate dominance.[19]

Corollary 2 *Given (10), $R(v)$ is strictly decreasing on $v \in (0, \overline{v})$ and bounded above by $F_w(v)/F_s(v)$.*

Proof. Keeping in mind that $b_w(v) > b_s(v)$, the Corollary follows directly from the relationship between $R(v)$ and bids. ■

3.3 Preferences

Having examined bidding strategies, we now turn to a related point, namely preferences. Specifically, we consider buyers' preferences for different auction formats, as well as for the type of competition they face.

Assuming there are only two buyers, and maintaining the assumption that (10) holds, Maskin and Riley (2000a) also show that the strong buyer prefers the second price auction to the first price auction, but that the weak buyer feels the other way. In fact, this follows easily from Proposition 3. The reason is that it implies that $q_w(v) > F_s(v)$ and $q_s(v) < F_w(v)$, meaning that the weak buyer wins more often in the first price auction than in the second price auction, and that the opposite holds for the strong buyer. The result then follows from (3).[20]

Corollary 3 *If $n = 2$ and (10) holds, the strong buyer strictly prefers the second price auction to the first price auction, while the weak buyer strictly prefers the first price auction to the second price auction.*

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[19] When it comes to comparing winning probabilities, $b_w(v) > b_s(v)$ implies that

$$\int_0^v \frac{q_s(x)}{q_s(v)} dx > \int_0^v \frac{q_w(x)}{q_w(v)} dx \quad \text{for all } v \in (0, \overline{v}),$$

which has a flavor of second order stochastic dominance with respect to the *conditional* distributions, e.g. $q_s(x)/q_s(v)$.

[20] This argument also proves that the result extends to the case where there are more than one weak buyer and one strong buyer (symmetric buyers must use symmetric equilibrium strategies). Likewise, if there are more buyers, and these can be arranged according to (10), the weakest buyer prefers the first price auction, and the strongest buyer the second price auction.

Maskin and Riley (2000a) claim that the ranking holds for buyers with high valuations even if (10) is replaced with the weaker assumption of first order stochastic dominance.[21] The fact that $q_s(v) = q_s^w(v) > F_s(v)$ can be utilized to prove this.

Corollary 4 *If $n = 2$ and $F_s(v) < F_w(v)$ for all $v \in (0, \overline{v})$, buyer s with valuation $\overline{v}$ strictly prefers the second price auction to the first price auction, while the opposite holds for buyer w with valuation $\overline{v}$. Moreover, $\overline{b}$ is between the expected valuation of buyer w and the expected valuation of buyer s.*

Proof. Since $q_s(v) > F_s(v)$, it follows that

$$EU_w(\overline{v}) = EU_s(\overline{v}) = \int_0^{\overline{v}} q_s(x)dx > \int_0^{\overline{v}} F_s(x)dx,$$

where the term furthest to the right equals the expected payoff to a weak buyer with valuation $\overline{v}$ in a second price auction. Since this buyer prefers the second price auction, his expected payment, namely the expected valuation of the strong buyer, must be less than his expected payment in the first price auction, where it is $\overline{b}$ (with a valuation of $\overline{v}$ he wins either auction with probability one). Furthermore, the second part of Lemma 1 implies that

$$EU_s(\overline{v}) = EU_w(\overline{v}) = \int_0^{\overline{v}} q_w(x)dx < \int_0^{\overline{v}} F_w(x)dx,$$

where the last term equals the expected payoff to a strong buyer with valuation $\overline{v}$ is a second price auction. An argument similar to that given before leads to the conclusion that $\overline{b}$ exceeds the expected valuation of the weak buyer. ■

We conclude this section with another result that also concerns the preferences of the buyers. As we have seen, the weak buyer may be more aggressive than the strong buyer. This raises the question of whether the strong buyer is better off facing a weak, but aggressive buyer, rather than another strong buyer, with the same distribution function as himself. Despite the aggressiveness, we show next that both a weak and a strong buyer prefer facing a weak rather than a strong buyer. This result does not rely on (10).

[21] See footnote 16 in Maskin and Riley (2000a).

Corollary 5 *Assume that $n = 2$, that $F_s(v) < F_w(v)$, for all $v \in (0, \overline{v})$, and that buyer i draws his valuation from either F_s or F_w. Then, buyer i with valuation v, $v \in (0, \overline{v}]$, is strictly better off if buyer j, $j \neq i$, draws his valuation from F_w rather than F_s.*[22]

Proof. First, assume that buyer i is strong, $F_i = F_s$. Then, we have already established that $q_i^j(v) > F_s(v)$, for all $v \in (0, \overline{v})$, if $F_j = F_w$. On the other hand, $q_i^j(v) = F_s(v)$, for all $v \in (0, \overline{v})$, if $F_j = F_s$, which follows from the fact that symmetric buyers bid symmetrically. Hence, for all $v \in (0, \overline{v})$, buyer i wins more often in equilibrium if buyer j is weak rather than strong. By (3), he is strictly better off facing a weak buyer.

Next, consider the possibility that buyer i is weak, $F_i = F_w$. To begin, assume buyer j is strong, $F_j = F_s$. In this case, buyer i's winning probability, $q_w(v)$, is less than $F_w(v)$ (Lemma 1). On the other hand, if buyer i were to face another weak buyer, $F_j = F_w$, he would win with probability $F_w(v)$. Again, the result follows from (3). ■

This result is related to a result in Lebrun (1998), who shows that if one buyer becomes much stronger, the other buyer is worse off. That is, Lebrun (1998) compares distributions that satisfy (10), rather than just first order stochastic dominance.

4 Predictable and unpredictable buyers

With the exception of Fibich. et al. (2002), we are not aware of any work which studies bidding strategies in asymmetric first price auctions without the assumption of first order stochastic dominance.[23] In this section, we illustrate that the approach proposed in this paper can be used to uncover new results under other forms of asymmetry.

Specifically, we assume there are exactly two buyers participating in the auction. One buyer, buyer p, is "predictable", whereas the other buyer, buyer u, is "unpredictable". Formally, we assume that F_u *is a mean preserving*

[22] This result also holds in the second price auction. It can also be extended to the case with $n-1$ buyers of the same kind (weak or strong), who would prefer the remaining buyer to be weak.

[23] Landsberger et. al. (2001) depart from the standard auction model, examining first price auctions when the ranking of valuations is common knowledge.

spread over F_p, or

$$\int_0^v F_u(x)dx > \int_0^v F_p(x)dx \ \forall v \in (0, \overline{v}), \ \int_0^{\overline{v}} F_u(x)dx = \int_0^{\overline{v}} F_p(x)dx. \quad (11)$$

(11) implies that the two buyers have the same expected valuation, but that buyer p is in some sense more "predictable" than buyer u.[24] Hence, this type of asymmetry can be used to describe situations where it is more difficult to "guess" the valuation of buyer u than of buyer p. Notice that in order for (11) to hold, F_p and F_u must cross.

It is worthwhile pointing out that the extension to predictable and unpredictable buyers is a natural next step once first price auctions with strong and weak buyers are well understood. After all, the distinction between strong and weak buyers boiled down to first order stochastic dominance of the distribution functions, whereas we are now considering second order stochastic dominance.[25]

Two observations follow easily from (11) and the preceding analysis. First, it is impossible for one buyer to be consistently (for all valuations) better off than another. If, for example, buyer u is consistently better off than buyer p, Lemma 1 would imply that $q_u > F_u$ and $q_p < F_p$, and

$$EU_u(\overline{v}) = \int_0^{\overline{v}} q_u(x)dx > \int_0^{\overline{v}} F_u(x)dx = \int_0^{\overline{v}} F_p(x)dx > \int_0^{\overline{v}} q_p(x)dx = EU_p(\overline{v}),$$

which contradicts (5). A similar argument shows that buyer p cannot be consistently better off than buyer u either. In contrast, in a *second price auction*, buyer p would be consistently better off than buyer u. Since $q_p = F_u$ and $q_u = F_p$ in a second price auction, the claim follows from (11). Hence, the auction format influences which buyer is better off.[26]

Second, it is also impossible for one buyer to bid consistently higher than another. Again, if buyer u bids consistently higher than p, then $q_u > F_p$ and $q_p < F_u$, and we obtain another contradiction of the same type as above.

[24] For instance, it is easily shown that F_u has a higher variance than F_p. Likewise, F_u has more mass at the tails than F_p.

[25] The first half of (11) implies that F_p second order stochastically dominates F_u. The somewhat limiting second half (which implies identical means) is unimportant in many of the results to follow.

[26] On the other hand, if we are comparing strong and weak buyers, as in Section 3, the strong buyer is strictly better off than the weak buyer in *both* auctions. He is better off in a second price auction since he wins more often than the weak buyer, and Theorem 1 shows that he is also better off in a first price auction.

To provide more details on payoff, strategies, and preferences, we impose an additional regularity assumption. Specifically, *we assume that* $F_p(v)/F_u(v)$ *is single-peaked*. In particular, we assume there is an interior point, x, such that F_p/F_u is strictly increasing to the left of x, and strictly decreasing to the right of x. The assumption implies that there is a unique interior point, v_c, such that $F_p(v_c) = F_u(v_c)$.[27] Figure 3 illustrates. This assumption is the counterpart to (10), which implied that $F_w(v)/F_s(v)$ has no peaks in the interior.

Proposition 4 *Given* $F_p(v)/F_u(v)$ *is single-peaked and satisfies (11), there exists* $v_1, v_2 \in (0, \overline{v})$ *such that (i)* $b_u(v) > b_p(v)$ *if* $v \in (0, v_1)$, $b_p(v) > b_u(v)$ *if* $v \in (v_1, \overline{v})$, *and (ii)* $EU_p(v) > EU_u(v)$ *if* $v \in (0, v_2)$, $EU_u(v) > EU_p(v)$ *if* $v \in (v_2, \overline{v})$ *and with* $v_2 < v_c < v_1$.

Proof. Define $\widehat{R}(v) \equiv EU_u(v)/EU_p(v)$. Using the same arguments as in Section 3, $\widehat{R}(v) > F_p(v)/F_u(v)$ if, and only if, $b_u(v) > b_p(v)$, in which case $\widehat{R}(v)$ is strictly increasing. Likewise, $\widehat{R}(v) < F_p(v)/F_u(v)$ if, and only if, $b_u(v) < b_p(v)$, in which case $\widehat{R}(v)$ is strictly decreasing. First, there can be no interior point, v, at which $\widehat{R}(v) \leq \min\{F_p(v)/F_u(v), 1\}$, because it would imply that $\widehat{R}(v)$ decreases to the right of v, making is impossible for $\widehat{R}(\overline{v}) = 1$. Hence, for all $v \in (0, v_c]$, $\widehat{R}(v) > F_p(v)/F_u(v)$ or $b_u(v) > b_p(v)$. It is easy to see that $\widehat{R}(v)$ must cross $F_p(v)/F_u(v)$ exactly once to the right of v_c, after which point $b_p(v) > b_u(v)$. Second, once $\widehat{R}(v)$ exceeds 1, it must remain above 1 (or the usual contradiction would arise). However, it is not possible for $\widehat{R}(v)$ to globally exceed 1, as we have already argued that neither buyer is consistently better off. Hence, $\widehat{R}(v)$ must start below 1, and eventually cross 1 from below. However, this is only possible to the left of v_c. Figure 3 depicts the equilibrium. ∎

Proposition 4 allows us to comment on both the distribution of bids, as well as preferences among auctions. Recall that with strong and weak buyers, the strong buyer's strength carried over into his bid (Proposition 2). A similar situation arises with predictable and unpredictable buyers.

Corollary 6 *The unpredictable buyer submits bids that are more unpredictable, in the sense that his bid is more likely to be extreme (the distribution has more mass at the tails).*

[27] An example which satisfies the assumption: $f_p(v) = 1$ (uniform), $f_u(v) = \frac{1}{2} + 2|v - \frac{1}{2}|$, with $\overline{v} = 1$.

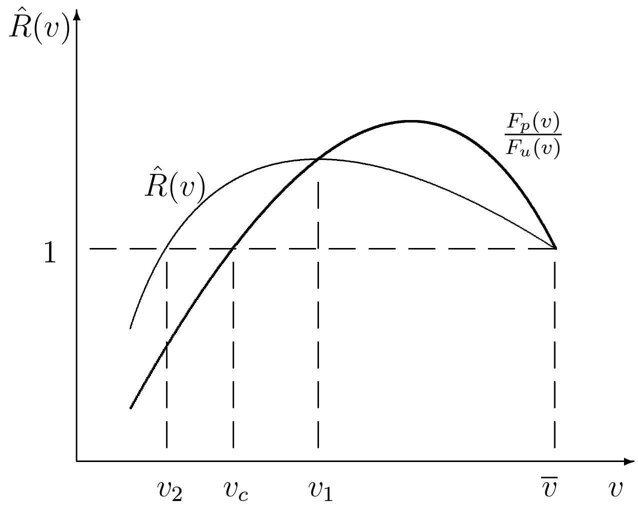

Figure 3: Single-peakedness of $F_p(v)/F_u(v)$.

Proof. Since $EU_p(v) > EU_u(v)$ for low valuations, it is possible to show, as in Lemma 1, that $q_u(v) < F_u(v)$ initially. Hence, $\Pr(p \text{ bids } \textit{below } b_u(v)) = q_u(v) < F_u(v) = \Pr(u \text{ bids } \textit{below } b_u(v))$. On the other hand, if v is high, then $EU_u(v) > EU_p(v)$, and $q_u(v) > F_u(v)$, or $\Pr(p \text{ bids } \textit{above } b_u(v)) = 1 - q_u(v) < 1 - F_u(v) = \Pr(u \text{ bids } \textit{above } b_u(v))$. ■

Finally, with weak and strong buyers it was possible to rank the two auction, either globally (Corollary 3) or at least at the top (Corollary 4). Here, with predictable and unpredictable buyers, we are able to rank auctions at the bottom, i.e. for low valuations.

Corollary 7 *If v is sufficiently low, buyer u prefers a first price auction to a second price auction, while the opposite holds for buyer p.*[28]

Proof. The Corollary follows from the fact that $b_u > b_p$ when v is small, implying that buyer u wins more often in a first price auction than in a second price auction, $q_u(v) > F_p(v)$. Likewise, buyer p prefers the second price auction, since he wins more often. ■

[28]Regarding a ranking for all valuations, at least one of the buyers will consistently prefer one auction to the other, but it is possible that the other buyer's preference depends on his valuation. Since $b_p > b_u$ if v is high, or $q_u < F_p$, the first price auction becomes less attractive relative to the second price auction for buyer u as the valuation increases (and vice versa for buyer p).

5 All-pay auctions

We now turn to a variation of the first price auction, the all-pay auction. In the all-pay auction, *every buyer pays his own bid whether or not he wins*, and the good is awarded to the highest bidder.[29] Hence, payoff is

$$EU_i(v) = vq_i(v) - b_i(v), \tag{12}$$

or, as before, (3). The equilibrium bid is therefore

$$b_i(v) = vq_i(v) - \int_0^v q_i(x)dx. \tag{13}$$

With two buyers, Amann and Leininger (1996) show that a unique equilibrium exists. Although it is possible that exactly one of the buyers bids zero for an interval of valuations, the bidding strategies are otherwise strictly increasing and differentiable, and the highest bid is the same for the two buyers. This seems to be all that is known about the all-pay auction with asymmetric buyers and incomplete information.

However, the approach proposed in this paper can be applied to the all-pay auction. In particular, since the highest bid is the same for the two buyers, (5) remains valid, as does Proposition 1.

Focussing on two buyers, one strong and one weak, most of the results in Section 3 are unchanged. As before, Theorem 1 holds for the all-pay auction as well.[30] This reveals that if one of the buyers bids zero for an interval of valuations, this buyer must be the weak buyer.[31,32] Intuitively, a weak buyer with a low valuation, knowing he faces strong competition, may simply be unwilling to sink the cost of bidding by participating in the auction. Hence, the winning probabilities are as in Figure 2 (when there is no mass of zero bids), or as in Figure 4 (with the weak buyer submitting a zero bid for several valuations).[33]

[29] The bid can be interpreted as the amount of (costly) effort expended in a tournament. Whether or not the buyer wins, the effort is sunk. Hence, the all-pay auction can be used to model lobbying activity, patent races, and the like.

[30] In particular, the proof in footnote 10 is unchanged.

[31] The buyer who bids zero for a mass of valuations earns zero surplus for such valuations, while the other buyer earns strictly positive surplus.

[32] Amann and Leininger (1996) present an example in which the weak buyer bids zero for an interval of valuations, but they do not provide any generalization of this.

[33] In the latter case, notice that q_s is discontinuous (but strictly positive) at $v = 0$.

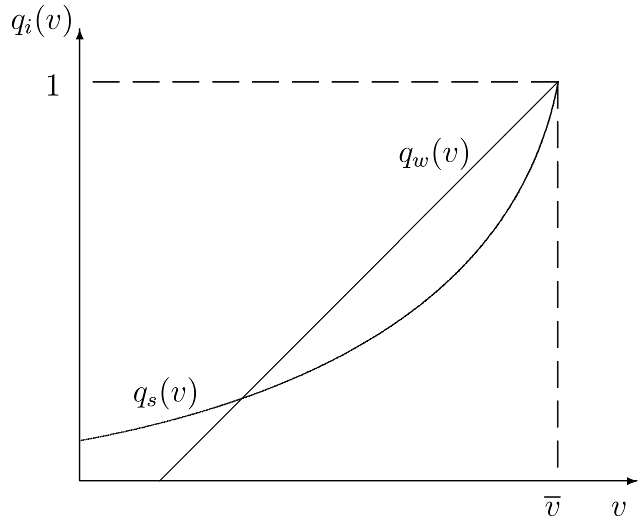

Figure 4: Winning probabilities in an all-pay auction.

Given the fact that Theorem 1 remains valid, it follows that Proposition 2 as well as Corollaries 1, 4, and 5 also hold in the all-pay auction.[34] Although the weak buyer bids more aggressively for some valuations (certainly those where $q_w(v) > q_s(v)$), it is clear that the weak buyer does not in general bid more aggressively for all valuations, since he may bid zero for a mass of valuations. Notice also that whenever the weak buyer bids zero for low valuations, he would prefer a second price auction (where he would win more often) if his valuation is low, but a first price auction if his valuation is high (Corollary 4). The opposite holds for the strong buyer.

While Amann and Leininger (1996) consider only the case of two buyers, Athey (2001) shows that a (pure strategy) equilibrium exists with more buyers, but does not discuss other properties of the equilibrium. With more buyers, however, it is not possible to rule out that more than one buyer, though not all, submit a bid of zero for a mass of valuations. Hence, the strong and weak buyer may both have zero probability of winning for a mass of valuations, if, for example, they face a very strong buyer. Thus, although the strong buyer is never worse off than the weak buyer, he is not necessarily always *strictly* better off when we allow for more buyers.

[34] Again, $q_s > F_s$ because $EU_s(v) = vq_s(v) - b_s(v) > EU_w(v) \geq vF_s(v) - b_s(v)$.

6 Conclusion

In this paper we offered a new approach to the analysis of asymmetric first price and all-pay auctions. Rather than looking at the system of differential equations determining equilibrium bidding strategies, we focused on the winning probabilities. This led to new insights, in particular regarding the interplay between the incentive for a strong buyer to shade his bid more, and his inherent strength. The method allowed us to suggest new, possibly simpler, proofs of existing results on bidding strategies, due to the fact that once winning probabilities are known, bidding strategies can be inferred. As a consequence of the link between winning probabilities and payoff, we also proved existing results regarding preferences among auction formats and the kind of competition a given buyer faces.

In the second half of the paper, the capability of the approach to uncover new results was demonstrated by considering a new type of asymmetry in first price auctions, as well as a different auction format, namely the all-pay auction.

The first price auction and the all-pay auction share the common features that a buyer with valuation $\overline{v}$ will win with probability one, and that his expected payment (his bid) is the same regardless of his identity. While the first property also holds in the second price auction and the optimal auction, among others, the second property generally does not hold in these auctions. It follows that the approach taken in this paper does not work equally well in all auctions.[35]

[35] Nevertheless, the second price auction and the optimal auction are well understood, as they are won by the bidder with the highest valuation or highest virtual valuation, respectively.

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