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Abstract

Timed contact algebras constitute an approach to a temporal version of a region based theory of space. In the general theory the underlying model of time does not have any structure, i.e. time in neither ordered nor required to be discrete or continuous. In this paper we want to investigate contact structures with a betweenness relation on points in time. Furthermore, we are interested in the relationship of a continuity axiom, an axiom of construction, and the fine structure of the time relation.

1. Introduction

The origins of mereotopology go back to the works of [6] on mereology and, on the other hand, the works of [4], [7], and [9] who used regions instead of points as the basic entity of geometry. In this "pointless geometry", points are second order definable as sets of regions, similar to the representation of Boolean algebras, where elements can be recovered as ultrafilters. Whitehead's addition to the mereological structures of Leśniewski (which were, basically, complete Boolean algebras B without a smallest element) was a "connection" (or "contact") relation C among nonempty regions, which, in its simplest form is a reflexive and symmetric relation satisfying an additional extensionality axiom. Historically, standard (models for) mereotopological structures were collections of regular closed (or regular open) sets of topological spaces $\langle X, \tau\rangle$ with the *standard (Whiteheadian) contact* among regions defined by

$$uCv \iff u \cap v \neq \emptyset.$$

The primary example is the collection of all nonempty regular closed sets of the Euclidean plane.

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The simplest algebraic counterpart of mereotopology are Boolean contact algebras, appearing in different papers under various names, which are Boolean algebras extended with the contact relation C satisfying some axioms. The elements of the Boolean algebra symbolize regions while the contact relation C corresponds to the additional topological aspect. For a recent overview of the topic we invite the reader to consult [8].

Timed contact algebras add a notion of time to those Boolean contact algebras, i.e. they constitute an adequate theory for regions moving in time. Those regions are modeled by a collection of snapshots in a static world. Given a set of points in time T a moving region is just a function from T to the static world. In other words, a region is described by a collection of static regions at any point in time t - its (spatial) extent at time t. We want to illustrate this idea by three examples. Notice that not all regions may be 'visible' at any given time; this situation is modeled by the fact that such a region is not in contact with anything, including itself, at that instance in time.

Example 1. We choose the Euclidean plane as static world and the set $\mathbb{R}^{\geq 0}$ of the non-negative real numbers as the model of time. Now, a moving region is a function from $\mathbb{R}^{\geq 0}$ to the regular closed sets of the plane. For example, r is the region so that $r(x)$ is the closed disc with radius 1 at the point (x, x). This region starts at the origin and moves along the diagonal without changing its shape. Another example is the region s where $s(x)$ is the closed disc at the origin with radius x. This region starts as the empty region and grows constantly bigger. It does not move at all.

Example 2. The second example is finite. Consider the set $X = \{1, 2, 3, 4, 5, 6\}$ with a contact structure based an the following picture

a	b	c
d	e	f

i.e. a is in contact with b and d, b is in contact with a, c and e etc. We say that two subsets A, B of X are in contact (in symbols: ACB) if and only if there is an $a \in A$ and an $b \in B$ which are in contact. This structure - the powerset $\mathcal{P}(X)$ of X together with the relation C - establishes a (finite) Boolean contact algebra. $\mathcal{P}(X)$ serves as the static world in our example, Time will be discrete, and we choose $T = \{0, 1, 2, 3\}$. Similar to the previous example a moving region is a function from T to $\mathcal{P}(X)$. For example, r defined by

$$r(0) = \{b\}, \;\; r(1) = \{a\}, \;\; r(2) = \{d\}, \;\; r(3) = \{e\}$$

is the region that starts at b and moves to e via a and d. Also in this example it is possible for regions to grow or shrink. The region s defined by

$$s(0) = \{b\}, \;\; s(1) = \{b, c\}, \;\; s(2) = \{c\}, \;\; s(3) = \emptyset$$

starts at b grows to $\{b, c\}$ and finally disappears.

Example 3. In general, a region at rest, i.e. a region that does not move (or change its shape), is modeled by a function that maps every point in time to the same static region. In this example we want to demonstrate the static world can be different at different points in time. In particular, there might be static regions available at some point in time that do not exist at others. Therefore, let $\mathbf{2} = \{0, 1\}$ and $\mathbf{4} = \{0, a, a^*, 1\}$ be the (finite) Boolean algebras with two and four elements, respectively. The contact on relation on both algebras is given by overlap, i.e. regions a and b are in contact if the have a nonzero meet or, more formally, aCb if and only if $a \cdot b \neq 0$. As model of time we choose the discrete set $\{0, 1\}$. Now, consider all function $f : \{0, 1\} \to \mathbf{2} \cup \mathbf{4}$ with $f(0) \in \mathbf{2}$ and $f(1) \in \mathbf{4}$. In this case there is no non-moving region that has a as its extent at time 1.

2. Contact Algebras

For any set X and $Y \subseteq X$ we denote by $X - Y$ the complement of Y in X. If X is clear from the context we simply write $-Y$. For a binary relation R on D, and $x \in D$, we let $R(x) = \{y : xRy\}$, the *range of x with respect to R.*

If $\sim$ is an equivalence relation on a set A, we will denote the equivalence class of an element $a \in A$ by $[a]$, i.e. $[a] = \{x \in A \mid a \sim x\}$.

2.1. Contact/Pre-Contact Relations

Throughout this paper, $(B, 0, 1, +, \cdot, *)$ is a Boolean algebra; if no confusion can arise we sometimes denote algebras by their base set. For details on Boolean algebras not explicitly mentioned in this paper we refer to [5].

In the remainder we will also use difference $a \setminus b := a \cdot b^*$ and the symmetric difference $a \triangle b := (a \setminus b) + (b \setminus a)$ of two elements a and b.

A binary relation C on a Boolean algebra B is called a *contact relation* (CR) if it satisfies:

C0. $(\forall a) 0(-C)a$;

C1. $(\forall a)[a \neq 0 \Rightarrow aCa]$;

C2. $(\forall a)(\forall b)[aCb \Rightarrow bCa]$;

C3. $(\forall a)(\forall b)(\forall c)[(aCb \text{ and } b \leq c) \Rightarrow aCc]$;

C4. $(\forall a)(\forall b)(\forall c)[aC(b + c) \Rightarrow (aCb \text{ or } aCc)]$.

The pair $\langle B, C\rangle$ is called a *Boolean Contact Algebra* (BCA).

Axioms C0 and C1 imply that 0 is the only element with $C(0) = \emptyset$, i.e. that is not connected to any element. This axiom is motivated in the static world, i.e. in contact structures without any notion of time, by the fact that we are not able to distinguish the empty region and a region that is not in contact to anything. In a timed environment this might not longer be true. A region in time might disappear, i.e. in our interpretation it is equal to the empty region, for a certain amount of time. To model this we use a weaker relations that are weaker than contact relations: A relation C on a Boolean algebra B is called a *pre–contact relation* if it satisfies C0,C2–C4 and

P1. $(\forall a)[(\exists b)aCb \Rightarrow aCa]$;

Any contact relation is a pre–contact relation. Conversely, the set $I := \{a \mid C(a) = \emptyset\}$ is a Boolean ideal for every pre–contact relation. We denote the congruence induced by I by $\sim_C$. The following lemma was shown in [3].

Lemma 1 *Let C be a pre–contact relation on a Boolean algebra B. Then:*

1. *I is an ideal and the relation C_I on B/I defined by*

$$[a]C_I[b] :\Longleftrightarrow aCb$$

is a contact relation.

2. *$a \sim_C b$ if and only if $C(a \triangle b) = \emptyset$.*

Notice that the range of C distributes over $+$ for any (pre-) contact relation, i.e.

$$\begin{aligned} x \in C(a+b) &\iff (a+b)Cx \\ &\iff aCx \text{ or } bCx \qquad \text{by C2-4} \\ &\iff x \in C(a) \cup C(b). \end{aligned}$$

The following property will be used later.

Lemma 2 *Let C be a pre–contact relation on a Boolean algebra B. Then $a \sim_C b$ implies $C(a) = C(a \cdot b) = C(b)$.*

Proof. Let $a \sim_C b$. Then $C(a \setminus b) \subseteq C(a \triangle b) = \emptyset$, and therefore,

$$\begin{aligned} C(a) &= C(a \cdot b + (a \setminus b)) \\ &= C(a \cdot b) \cup C(a \setminus b) \\ &= C(a \cdot b). \end{aligned}$$

The property $C(b) = C(a \cdot b)$ is shown analogously. $\square$

A (pre)-contact relation is called *extensional* if and only if it satisfies the following axiom:

C5. $(\forall a)(\forall b)[C(a) = C(b) \Rightarrow a = b]$.

Notice that an extensional pre–contact relation is already a contact relation.

2.2. Timed Contact Algebras

A timed contact structure was first defined in [3], and is given by a set modeling time, a Boolean algebra of regions, and a family of pre–contact relations.

Definition 1 *We call $\langle T, B, (C_t)_{t \in T} \rangle$ a timed contact algebra if T is a non-empty set, B Boolean algebra, and C_t is a pre–contact relation on B for every $t \in T$.*

The relationship aC_tb denotes the fact that the regions a and b are in contact at time t.

Notice that the family of pre–contact relations can be seen as a ternary relation $C \subseteq T \times B \times B$ defined by $C = \bigcup_{t \in T} \{t\} \times C_t$. In the remainder of the paper we will adapt that view even though we will not change the notation.

It is easy to verify that the set of function B^T from an arbitrary set T to a Boolean (pre)-contact algebra B is a timed contact algebra. In general, we cannot guarantee that the contact structures at each point in time are the same (or are isomorphic). We will show below that every faithful timed contact algebra can be represented by a subdirect product.

For every $t \in T$ the pre–contact relation C_t induces a congruence relation on B which we denote by $\sim_t$. Furthermore, we denote the quotient algebra by B_t and the natural homomorphism by $\pi_t : B \to B_t$.

We call a timed contact algebra *faithful* if and only if it satisfies the following:

FAT. $(\forall a)(\forall b)[(\forall t)(a \sim_t b) \Rightarrow a = b]$.

The next lemma provides an algebraic characterization of faithfulness:

Lemma 3 *A timed contact algebra $\langle T, B, (C_t)_{t \in T} \rangle$ is faithful if and only if the mapping $\pi : B \to \prod_{t \in T} B_t$ defined by $\pi(b)_t = \pi_t(b)$, i.e. the t-component of $\pi(b)$ is $\pi_t(b)$, is a subdirect embedding.*

Proof. It is well known that π is an embedding if and only if the family $(\pi_t)_{t \in T}$ separates points (see e.g. [1]), and the latter property is just FAT. Since each B_t is a quotient of B, π is a subdirect embedding. $\square$

As a straightforward generalization of C5 we call a timed contact algebra *extensional* if and only if it satisfies the following condition:

EXT. $(\forall a)(\forall b)[(\forall t)(C_t(a) = C_t(b)) \Rightarrow a = b]$.

From Lemma 2 we immediately conclude that every extensional timed contact algebra is also faithful.

2.3. Topological models

First we recall some notions from topology. By a topological space (X, τ) we mean a set X provided with a family τ of subsets, called open sets, which contains the empty set and the whole set X, and is closed with respect to arbitrary unions and finite intersections. A subset $a \subseteq X$ is called *closed* if it is the complement of an open set.

In every topological space one can define the following operations on subsets $a \subseteq X$:

1. $Int(a) = \bigcup\{o \in \tau \mid o \subseteq a\}$ (the interior of a), i.e., the union of all open sets contained in a.
2. $Cl(a) = \bigcap\{c \text{ is closed} \mid a \subseteq c\}$ (the closure of a), i.e., the intersection of all closed sets containing a.

Cl and Int are interdefinable, i.e. $Cl(a) = -Int(-a)$ and $Int(a) = -Cl(-a)$.

A subset a of X is called *regular closed* if $Cl(Int(a)) = a$. We denote by $RC(X)$ the family of regular closed sets of X. It is a well known fact that $RC(X)$ is a Boolean algebra with respect to the following operations and constants:

$$0 = \emptyset, 1 = X, a + b = a \cup b \text{ and } a \cdot b = Cl(Int(a \cap b)).$$

$RC(X)$ naturally provides a contact relation C defined by aCb if and only if $a \cap b \neq \emptyset$. C is called *the standard* (or *Whiteheadian*) *contact relation* on $RC(X)$.

We regard a region in time as a function $f : T \to RC(X)$, i.e. the function providing the spatial extent of the region for every point in time.

We denote the set of all such function by X_T and define $C_t \subseteq X_T \times X_T$ by

$$fC_tg :\iff f(t) \cap g(t) \neq \emptyset.$$

Notice that for each $t \in T$ the relation C_t is defined by the standard contact relation on B, i.e. fC_tg if and only if $f(t)C^wg(t)$.

The following theorem was shown in [3].

Theorem 1 *Let $\langle T, B, (C_t)_{t\in T}\rangle$ be a timed contact structure. Then there is a compact and semi-regular T_0 space $\langle X, \tau\rangle$ and an embedding $k : B \to X_T$ with aC_tb if and only if $k(a)(t) \cap k(b)(t) \neq \emptyset$.*

3. Basis of Regions At Rest

In a concrete topological model we may call the constant functions *regions at rest* since they neither move nor change their shape. Since the elements in the abstract theory need not to be functions we cannot immediately transfer this idea. Our approach will be based on a binary relation and maximal cliques thereof.

We say that a region *a is at rest with respect to a region b* (or *a and b are at rest* or *a is not moving with respect to b*), denoted by aRb, if and only if

$$(\exists t)(aC_tb) \Rightarrow (\forall t)(aC_tb).$$

Definition 2 *Let $\langle T, B, (C_t)_{t\in T}\rangle$ be a timed contact structure. A subalgebra S of B is called static if and only if the elements of S are pairwise at rest, i.e. aRb for all $a, b \in S$.*

Lemma 4 *Each timed contact structure $\langle T, B, (C_t)_{t\in T}\rangle$ permits a non-empty and maximal static subalgebra.*

Proof. First, it is easy to verify that the subalgebra $\{0, 1\}$ is indeed static because $0(-C_t)1$ for any $t \in T$. The existence of a maximal subalgebra now follows from Zorn's Lemma: In order to apply that lemma we show that $S = \bigcup_{i\in I} S_i$ of a chain $(S_i)_{i\in I}$ (with respect to inclusion) of static subalgebras is again a static subalgebra. It is easy to verify that S is a subalgebra of B. Now, assume $a, b \in S$ and that there is a $t \in T$ with aC_tb. Then there is a $j \in I$ with $a, b \in S_j$ since $(S_i)_{i\in I}$ is a chain. From the fact that S_j is a static subalgebra we conclude aC_tb for all $t \in T$. $\square$

Maximal static subalgebras are usually not unique. Furthermore, such a maximal static subalgebra might not be rich enough to characterize the extent of every region at any point in time. To illustrate this, consider Example 3 again. The set of constant functions, i.e. the functions $\bar{0}(x) = 0$ and $\bar{1}(x) = 1$, is the universe of a maximal static subalgebra, but none of those functions satisfies $f(1) = a$.

Definition 3 *Let $\langle T, B, (C_t)_{t\in T}\rangle$ be a timed contact structure. A maximal static subalgebra S is called a* basis of regions at rest *(or simply* basis*) if and only if for all $b \in B$ and all $t \in T$ there is $a \in S$ with $b \sim_t a$. In this situation the structure $\langle T, B, (C_t)_{t\in T}, S\rangle$ is called a grounded (timed) contact algebra.*

Since elements of a basis are pairwise at rest, the restriction of any of the relations C_t to S yields the same relation. We will denote this relation by C_S, i.e. for all $a, b \in S$ we have

$$aC_Sb \iff (\exists t \in T)aC_tb \iff (\forall t \in T)aC_tb.$$

Theorem 2 *Let $\langle T, B, (C_t)_{t\in T}, S\rangle$ be a grounded contact algebra. Then $\langle S, C_S\rangle$ is*

1. *a BCA if B is faithful,*
2. *an extensional BCA if B is extensional.*

Proof.

1. It is sufficient to show the ideal $I = \{a \in S \mid C_S(a) = \emptyset\}$ just contains 0. Assume there $a \neq 0$ with $C_S(a) = \emptyset$. We want to show that $C_t(a) = \emptyset$ for all $t \in T$ since this implies $a \sim_t 0$ for all $t \in T$, and, hence, $a = 0$ using FAT, a contradiction. Therefore, assume that there is a $c \in B$ and a $t \in T$ with aC_tc. Then there is an element $b \in S$ with $b \sim_t c$ because S is a basis. We obtain $a \in C_t(c) = C_t(b)$ by Lemma 2, and, hence, aC_Sb, a contradiction.
2. Since extensionality implies injectivity S is a BCA by 1. Now, assume $C_S(a) = C_S(b)$ for $a, b \in S$. We want to show that $C_t(a) = C_t(b)$ for all $t \in T$ since this implies $a = b$ by EXT. Therefore, assume aC_tc. Then there is a $d \in S$ with $c \sim_t d$ since S is a basis. We get $a \in C_t(c) = C_t(d)$, i.e. aC_td. From $d \in S$ we conclude aC_Sd, and, hence, $d \in C_S(a) = C_S(b)$. This immediately implies $b \in C_t(d) = C_t(c)$, i.e. bC_tc. $\square$

In the next theorem we want to establish that a basis really captures regions at rest, i.e. that the elements in S can be regarded as constant functions.

Theorem 3 *Let $\langle T, B, (C_t)_{t \in T}, S\rangle$ be an faithful grounded contact algebra. Then B can be embedded into the algebra of functions S^T. Furthermore, the basis S corresponds to the set of constant functions in S^T.*

Proof. By Theorem 2 we know that S together with C_S is a BCA. We want to show that $c \sim_t a$ and $c \sim_t b$ for a $t \in T$ with $a, b \in S$ implies $a = b$, i.e. that the mapping $h_t : B \to S$ defined by $h_t(c) = a$ if and only if $c \sim_t a$ and $a \in S$ is well defined. Since $\sim_t$ is an equivalence relation we obtain $a \sim_t b$. From $a \sim_t b$ is equivalent to $C(a \triangle b) = \emptyset$ and $a, b \in S$ we get $a \sim_t b$ for all $t \in T$. Injectivity implies $a = b$. Now, define $h(c)(t) := h_t(c)$, and we have to show that cC_td if and only if $h_t(c)C_Sh_t(d)$. This follows immediately from $C_t(h_t(c)) = C_t(c) \ni d$ and $c \in C_t(d) = C_t(h_t(d))$.

It remains to show that h maps the elements of S to constant functions. Since $\sim_t$ is reflexive we have $a \sim_t a$, and, hence, $h_t(a) = a$ for all $t \in T$. Therefore, $h(a)(t) = a$ for all $a \in S$. □

Notice that the previous theorem can also be stated in terms of a subdirect product. It shows that in the current case all components of the product B_t are isomorphic to S.

4. Axiom of Construction

We are interested in ensuring the existence of a moving region if a finite snapshot is provided. This is of particular interest if we have a notion of static regions, i.e. a grounded contact structure. Assume one fixes a certain (finite) number of points in time, and for each such point a static region. Is there a region that moves through those static regions at those points in time? The following axiom of construction ensures the existence of such a region:

CONS. $(\forall a)(\forall b)(\forall t_1)(\forall t_2)[t_1 \neq t_2 \Rightarrow (\exists c)(c \sim_{t_1} a \text{ and } c \sim_{t_2} b)]$.

Even though the Axiom of Construction is motivated by regions at rest, the axiom does not require the existence of a basis.

5. Betweenness Relation

Betweenness relations were studied in detail in [2]. A ternary relation Btw on a set U is called a betweenness relation if it satisfies

BT0. $(\forall a)\,\mathrm{Btw}(a, a, a)$;

BT1. $(\forall a)(\forall b)(\forall c)[\mathrm{Btw}(a, b, c) \Rightarrow \mathrm{Btw}(c, b, a)]$;

BT2. $(\forall a)(\forall b)(\forall c)[\mathrm{Btw}(a, b, c) \Rightarrow \mathrm{Btw}(a, a, b)]$;

BT3. $(\forall a)(\forall b)(\forall c)[(\mathrm{Btw}(a, b, c) \text{ and } \mathrm{Btw}(a, c, b)) \Rightarrow b = c]$.

It was shown in [2] that a betweenness relation is generated by a partial order if and only if it satisfies the additional axioms:

BT4. There are no odd cycles in the graph of the relation $\mathrm{Btw}(a, a, b)$;

BT5. $(\forall a)(\forall b)(\forall c)[(\mathrm{Btw}(a, b, c) \text{ and } \mathrm{Btw}(b, c, d) \text{ and } b \neq c) \Rightarrow \mathrm{Btw}(a, b, d)]$.

Notice that BT4 can neither be expressed by a finite set of first-order axioms nor by using a finite number of variables. Axiom BT5 is also called "outer transitivity axiom".

We call a betweenness relation Btw on U *dense* if and only if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$ there is a $u \in U$ with $u \neq u_1, u \neq u_2$ and $\mathrm{Btw}(u_1, u, u_2)$.

Definition 4 *Let $\langle T, B, (C_t)_{t \in T}\rangle$ be a timed contact structure, and* Btw *be a betweenness relation on T. Then $\langle T, \mathrm{Btw}, B, (C_t)_{t \in T}\rangle$ is called a BLT-algebra.*

6. Axiom of Continuity

In order to state the axiom we need two more predicates. The ternary relation NZ defined by

$$NZ(t_1, t_2, a) :\Longleftrightarrow (\forall t)(\mathrm{Btw}(t_1, t, t_2) \Rightarrow a \not\sim_t 0)$$

indicates that the region a does not disappear at any time between t_1 and t_2. Furthermore, we are interested in timed version of external contact. In the static world external contact $aECb$ can be defined by aCb and $a \cdot b = 0$. Using the relation $\sim_t$ we define external contact at t by

$$aEC_tb :\Longleftrightarrow aC_tb \text{ and } a \cdot b \sim_t 0.$$

The Axiom of Continuity requires that two regions that

1. are connected at t_1,
2. disconnected at t_2, and
3. do not disappear between t_1 and t_2

must be in external contact at some point in time between t_1 and t_2. Formally, we define

$$\text{CONT.}\ (\forall a)(\forall b)(\forall t_1)(\forall t_2)(\exists t)[(NZ(t_1,t_2,a) \text{ and } NZ(t_1,t_2,b) \text{ and } aC_{t_1}b \text{ and } a(-C_{t_2})b) \Rightarrow (\mathrm{Btw}(t_1,t,t_2) \text{ and } a(EC_t)b)].$$

We obtain the following relationship between the temporal structure and the Axiom of Construction:

Lemma 5 *Let $\langle T, \mathrm{Btw}, B, (C_t)_{t\in T}, S\rangle$ be an extensional grounded BLT algebra satisfying* CONT, CONS *and $|S| > 2$. Then* Btw *is dense.*

Proof. Since $|S| > 2$ there is an element $a \in S$ with $a \neq 0, 1$. By Theorem 2 $\langle S, C_S\rangle$ is an extensional BCA, i.e. there is $b \in S$ with $b \neq 0$ and $a(-C_S)b$.

From $a \neq 0$ we conclude that $C_S(a) \neq \emptyset$ using the extensionality of C_S. This is equivalent to aC_ta for all $t \in T$. The property bC_tb for all $t \in T$ follows analogously.

Now, assume $t_1, t_2 \in T$ with $t_1 \neq t_2$. CONS implies that there is a region $c \in B$ with $c \sim_{t_1} a$ and $c \sim_{t_2} b$. We distinguish two cases:

$-NZ(t_1,t_2,c)$**:** In this case there is a $t \in T$ with $\mathrm{Btw}(t_1,t,t_2)$ and $c \sim_t 0$, i.e. $c(-C_t)c$. From $a \sim_{t_1} c$ and $aC_{t_1}a$ we conclude $cC_{t_1}a$. This implies $c \not\sim_{t_1} 0$, and, hence, $t \neq t_1$. The property $t \neq t_2$ is shown analogously using $c \sim_{t_2} b$ and $bC_{t_2}b$.

$NZ(t_1,t_2,c)$**:** Since aC_ta for all $t \in T$ we get $NZ(t_1,t_2,a)$. Furthermore, we have $a \in C_{t_1}(a) = C_{t_1}(c)$, i.e. $aC_{t_1}c$, and $a \notin C_{t_2}(b) = C_{t_2}(c)$, i.e. $a(-C_{t_2})c$. From CONT we conclude that there is a $t \in T$ with $\mathrm{Btw}(t_1,t,t_2)$ and aEC_tc. Since $a \in C_{t_1}(a) = C_{t_1}(a \cdot c)$ we get $a(-EC_{t_1})c$. This verifies $t \neq t_1$. Since $a(-C_{t_2})b$ and $c \sim_{t_2} b$ we have $a(-C_{t_2})c$. This shows $t \neq t_2$. $\square$

7. Conclusion and Outlook

In this paper we started to investigate axioms that relate the existence of regions and the temporal structure of timed contact algebras. Our axiom of continuity serves as a first approximation of a 'smooth' movement of regions. Future work will concentrate on this aspect.

Furthermore, the spatial component of a timed contact algebra is based on a binary contact relation at each point in time. A temporal/modal logic based on a temporal and spatial modality for this kind of theory is of interest.

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