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1 Introduction

The motivation for this paper comes from the following sources. First, one can observe that the two major concepts underlying the methods of reasoning with incomplete information are the concept of degree of truth of a piece of information and the concept of approximation of a set of information items. We shall refer to the theories employing the concept of degree of truth as to theories of fuzziness and to the theories employing the concept of approximation as to theories of roughness (see [5] for a survey). The algebraic structures relevant to these theories are residuated lattices ([6], [11], [15], [16], [21], [22]) and Boolean algebras with operators ([18], [20], [9], [10]), respectively. Residuated lattices provide an arithmetic of degrees of truth and Boolean algebras equipped with the appropriate operators provide a method of reasoning with approximately determined information. Both classes of algebras have a lattice structure as a basis. Second, both theories of fuzziness and theories of roughness develop generalizations of relation algebras to algebras of fuzzy relations [19] and algebras of rough relations ([3], [4], [8]), respectively. In both classes a lattice structure is a basis. Third, not necessarily distributive lattices with modal operators, which can be viewed as most elementary approximation operators, are recently developed in [23] (distributive lattices with operators are considered in [13] and [24]). With this background, our aim is this paper is to begin a systematic study of the classes of algebras that have the structure of a (not necessarily distributive) lattice and, moreover,

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in each class there are some operators added to the lattice which are relevant for binary relations. Our main interest is in developing relational representation theorems for the classes of lattices with operators under consideration. More precisely, we wish to guarantee that each algebra of our classes is isomorphic to an algebra of binary relations on a set. We prove the theorems of that form by suitably extending the Urquhart representation theorem for lattices ([25]) and the representation theorems presented in [1]. The classes defined in the paper are the parts which put together lead to what might be called lattice-based relation algebras. Our view is that these algebras would be the weakest structures relevant for binary relations. All the other algebras of binary relations considered in the literature would then be their signature and/or axiomatic extensions.

Throughout the paper we use the same symbol for denoting an algebra or a relational system and their universes.

2 Doubly ordered sets

In this section we recall the notions introduced in [25] and some of their properties.

Definition 1. Let X be a non–empty set and let $\leqslant_1$ and $\leqslant_2$ be two partial orderings in X. A structure $(X, \leqslant_1, \leqslant_2)$ is called a ***doubly ordered set*** iff for all $x, y \in X$, if $x \leqslant_1 y$ and $x \leqslant_2 y$ then $x = y$. □

Definition 2. Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. We say that $A \subseteq X$ is $\leqslant_1$–***increasing*** (resp. $\leqslant_2$–***increasing***) whenever for all $x, y \in X$, if $x \in A$ and $x \leqslant_1 y$ (resp. $x \leqslant_2 y$), then $y \in A$. □

For a doubly ordered set $(X, \leqslant_1, \leqslant_2)$, we define two mappings $l, r : 2^X \to 2^X$ by: for every $A \subseteq X$,

$$l(A) = \{x \in X : (\forall y \in X)\ x \leqslant_1 y \Rightarrow y \notin A\} \tag{1}$$

$$r(A) = \{x \in X : (\forall y \in X)\ x \leqslant_2 y \Rightarrow y \notin A\}. \tag{2}$$

Observe that mappings l and r can be expressed in terms of modal operators as follows: $l(A) = [\leqslant_1](-A)$ and $r(A) = [\leqslant_2](-A)$, where $-$ is the Boolean complement and $[\leqslant_i]$, $i = 1, 2$, are the necessity operators determined by relations $\leqslant_i$. Consequently, r and l are intuitionistic–like negations.

Definition 3. Given a doubly ordered set $(X, \leqslant_1, \leqslant_2)$, a subset $A \subseteq X$ is called ***l–stable*** (resp. ***r–stable***) iff $l(r(A)) = A$ (resp. $r(l(A)) = A$). □

The family of all l-stable (resp. r–stable) subsets of X will be denoted by $L(X)$ (resp. $R(X)$).

Recall the following notion from e.g. [7]:

Definition 4. Let $(X, \leqslant_1)$ and $(Y, \leqslant_2)$ be partially ordered sets and let $f : X \to Y$ and $g : Y \to X$. We say that f and g are ***Galois connection*** iff for all $x, y \in X$

$$x \leqslant_1 g(y) \text{ iff } y \leqslant_2 f(x). \quad \Box$$

Lemma 1. [23] *For any doubly ordered set $(X, \leqslant_1, \leqslant_2)$ and for any $A \subseteq X$,*

- **(i)** *$l(A)$ is $\leqslant_1$–increasing*
- **(ii)** *$r(A)$ is $\leqslant_2$–increasing*
- **(iii)** *if A is $\leqslant_1$–increasing, then $r(A) \in R(X)$*
- **(iv)** *if A is $\leqslant_2$–increasing, then $l(A) \in L(X)$*
- **(v)** *if $A \in L(X)$, then $r(A) \in R(X)$*
- **(vi)** *if $A \in R(X)$, then $l(A) \in L(X)$*
- **(vii)** *if $A, B \in L(X)$, then $r(A) \cap r(B) \in R(X)$.* $\blacksquare$

Lemma 2. [23] *The family of $\leqslant_i$–increasing sets, $i = 1, 2$, forms a distributive lattice, where join and meet are union and intersection of sets.* $\blacksquare$

Lemma 3. [25] *For every doubly ordered set $(X, \leqslant_1, \leqslant_2)$, the mappings l and r form a Galois connection between the lattice of $\leqslant_1$–increasing subsets of X and the lattice of $\leqslant_2$–increasing subsets of X.* $\blacksquare$

In other words, Lemma 3 says that for any $A \in L(X)$ and for any $B \in R(X)$, $A \subseteq l(B)$ iff $B \subseteq r(A)$.

Lemma 4. *For every doubly ordered set $(X, \leqslant_1, \leqslant_2)$ and for every $A \subseteq X$,*

- **(i)** *$l(r(A)) \in L(X)$ and $r(l(A)) \in R(X)$*
- **(ii)** *if A is $\leqslant_1$–increasing, then $A \subseteq l(r(A))$*
- **(iii)** *if A is $\leqslant_2$–increasing, then $A \subseteq r(l(A))$.*

Proof. Direct consequence of Lemmas 1 and 3. $\blacksquare$

Lemma 4 immediately implies:

Corollary 1. *For every doubly ordered set $(X, \leqslant_1, \leqslant_2)$ and for every $A \subseteq X$,*

(i) *if* $A \in L(X)$*, then* $A \subseteq l(r(A))$
(ii) *if* $A \in R(X)$*, then* $A \subseteq r(l(A))$*.* ∎

Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. Define two binary operations in 2^X: for all $A, B \subseteq X$,

$$A \sqcap B = A \cap B \tag{3}$$

$$A \sqcup B = l(r(A) \cap r(B)). \tag{4}$$

Observe that $\sqcup$ is defined from $\sqcap$ resembling a De Morgan law with two different negations.
Moreover, put

$$\mathbf{0} = \emptyset. \tag{5}$$

$$\mathbf{1} = X \tag{6}$$

Lemma 5. [25] *For any doubly ordered set* $(X, \leqslant_1, \leqslant_2)$*, the system* $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ *is a lattice.* ∎

Definition 5. Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. The lattice $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ is called the ***complex algebra of*** X. □

3 Urquhart representation of lattices

In this paper we are interested in studying relationships between relational structures (frames) providing Kripke–style semantics of logics, and algebras based on lattices. Therefore, we do not assume any topological structure in the frames. As a result, we have a weaker form of the representation theorems than the original Urquhart result, which requires compactness.

Let $(W, \wedge, \vee, 0, 1)$ be a bounded lattice.

Definition 6. A ***filter-ideal pair*** of a lattice W is a pair $x = (x_1, x_2)$ such that x_1 is a filter of W, x_2 is an ideal of W and $x_1 \cap x_2 = \emptyset$. □

The family of all filter–ideal pairs of a lattice W will be denoted by $FIP(W)$.

Let us define the following two quasi ordering relations on $FIP(W)$: for any $(x_1, x_2), (y_1, y_2) \in FIP(W)$,

$$(x_1, x_2) \preccurlyeq_1 (y_1, y_2) \quad \text{iff} \quad x_1 \subseteq y_1 \tag{7}$$

$$(x_1, x_2) \preccurlyeq_2 (y_1, y_2) \quad \text{iff} \quad x_2 \subseteq y_2. \tag{8}$$

Next, define

$$(x_1, x_2) \preccurlyeq (y_1, y_2) \quad \text{iff} \quad (x_1, x_2) \preccurlyeq_1 (y_1, y_2) \ \& \ (x_1, x_2) \preccurlyeq_2 (y_1, y_2).$$

We say that $(x_1, x_2) \in FIP(W)$ is *maximal* iff it is maximal wrt $\preccurlyeq$. We will write $X(W)$ to denote the family of all maximal filter–ideal pairs of the lattice W.

Observe that $X(W)$ is a binary relation on 2^W.

Proposition 1. [25] *Let W be a bounded lattice. For any $(x_1, x_2) \in FIP(W)$ there exists $(y_1, y_2) \in X(W)$ such that $(x_1, y_1) \preccurlyeq (y_1, y_2)$.* ∎

For any $(x_1, x_2) \in FIP(W)$, the maximal filter–ideal pair (y_1, y_2) such that $(x_1, x_2) \preccurlyeq (y_1, y_2)$ will be referred to as an *extension* of (x_1, x_2).

Definition 7. Let $(W, \wedge, \vee, 0, 1)$ be a bounded lattice. The ***canonical frame of*** W is the structure $(X(W), \preccurlyeq_1, \preccurlyeq_2)$. □

Lemma 6. *For every bounded lattice W,*

- **(i)** *its canonical frame $(X(W), \preccurlyeq_1, \preccurlyeq_2)$ is a doubly ordered set*
- **(ii)** *for all $x, y \in X(W)$, if $x \preccurlyeq_1 y$ and $x \preccurlyeq_2 y$, then $x = y$.* ∎

Consider the complex algebra $(L(X(W)), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ of the canonical frame of a lattice $(W, \wedge, \vee, 0, 1)$. Observe that $L(X(W))$ is an algebra of subrelations of $X(W)$.

Let us define the mapping $h : W \to 2^{X(W)}$ as follows: for every $a \in W$,

$$h(a) = \{x \in X(W) : a \in x_1\}. \tag{9}$$

Theorem 1. [25] *For every lattice $(W, \wedge, \vee, 0, 1)$ the following assertions hold:*

- **(i)** *For every $a \in W$, $r(h(a)) = \{x \in X(W) : a \in x_2\}$*
- **(ii)** *$h(a)$ is l–stable for every $a \in W$*
- **(iii)** *h is a lattice embedding.*

Proof. By way of example we prove **(iii)**.

We show that h is injective. Assume that for some $a, b \in W$, $h(a) = h(b)$. It follows that for every $x \in X(W)$, $a \in x_1$ iff $b \in x_1$. In particular, if $x_1 = [a) = \{z \in W : a \leqslant z\}$, then clearly $a \in [a)$, and also by the assumption $b \in [a)$. Hence $a \leqslant b$. Similarly, if $x_1 = [b)$, then $b \leqslant a$. We conclude that $a = b$.
Now we show that h preserves the operations. By way of example we prove that $h(a) \sqcup h(b) = h(a \vee b)$. Indeed, for every $a, b \in W$,

$$
\begin{aligned}
&h(a) \sqcup h(b) \\
&\quad = l(r(\{x \in X(W) : a \in x_1\}) \cap r(\{x \in X(W) : b \in x_1\})) \\
&\quad = l(\{x \in X(W) : a \in x_2\} \cap \{x \in X(W) : b \in x_2\}) && \text{from } \mathbf{(i)} \\
&\quad = l(\{x \in X(W) : a \vee b \in x_2\}) && \text{since } x_2 \text{ is an ideal} \\
&\quad = lr(\{x \in X(W) : a \vee b \in x_1\}) && \text{from } \mathbf{(i)} \\
&\quad = lr(h(a \vee b)) && \text{the definition of } h \\
&\quad = h(a \vee b) && \text{from } \mathbf{(ii)}. \qquad \square
\end{aligned}
$$

The following theorem is a weak version of the Urquhart result.

Theorem 2 (Representation theorem for lattices). *Every bounded lattice is isomorphic to a subalgebra of the complex algebra of its canonical frame.* □

4 LC algebras

An LC algebra is a bounded lattice with an additional unary operator which is an abstract counterpart of the relational converse.

Definition 8. An ***LC algebra*** is a structure $(W, \wedge, \vee, 0, 1, ^{\smile})$ such that $(W, \wedge, \vee, 0, 1)$ is a bounded lattice and $^{\smile}$ is a unary operator on W such that for all $a, b \in W$,

(C.1) $a^{\smile\smile} = a$
(C.2) $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$. □

For an LC algebra W and any $a \in W$, $a^{\smile}$ is called a *converse of* a.

The following lemma gives the basic properties of the converse operation.

Lemma 7. *Let* $(W, \wedge, \vee, 0, 1, ^{\smile})$ *be an LC algebra. Then the following assertions hold:*

(i) $0^{\smile} = 0$, $1^{\smile} = 1$.
(ii) *for all* $a, b \in W$, $a \leqslant b$ *implies* $a^{\smile} \leqslant b^{\smile}$
(iii) *for all* $a, b \in W$, $(a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}$.

Proof. The proof of **(i)** is similar to the one presented in [2]. Namely, by Definition 8 we have: $0^{\smile} = 0 \vee 0^{\smile} = 0^{\smile\smile} \vee 0^{\smile} = (0^{\smile} \vee 0)^{\smile} = 0^{\smile\smile} = 0$. Analogously, $1 = 1 \vee 1^{\smile} = 1^{\smile\smile} \vee 1^{\smile} = (1^{\smile} \vee 1)^{\smile} = 1^{\smile}$.

(ii) Assume that $a \leqslant b$. Then $a \vee b = b$, so $(a \vee b)^{\smile} = b^{\smile}$. By axiom **(C.2)**, $a^{\smile} \vee b^{\smile} = b^{\smile}$, so $a^{\smile} \leqslant b^{\smile}$.

(iii) **($\leqslant$)** Let $a, b \in W$. Since $a \wedge b \leqslant a$, by **(ii)** we get $(a \wedge b)^{\smile} \leqslant a^{\smile}$. Similarly, $(a \wedge b)^{\smile} \leqslant b^{\smile}$, which yields $(a \wedge b)^{\smile} \leqslant a^{\smile} \wedge b^{\smile}$.

($\geqslant$) Since $a^{\smile} \wedge b^{\smile} \leqslant a^{\smile}$, we have $(a^{\smile} \wedge b^{\smile})^{\smile} \leqslant a^{\smile\smile} = a$ by **(ii)** and **(C.1)**. Analogously, $(a^{\smile} \wedge b^{\smile})^{\smile} \leqslant b$, so $(a^{\smile} \wedge b^{\smile})^{\smile} \leqslant a \wedge b$. Applying again **(C.1)** and **(ii)** we get $a^{\smile} \wedge b^{\smile} = (a^{\smile} \wedge b^{\smile})^{\smile\smile} \leqslant (a \wedge b)^{\smile}$. ■

Given an LC algebra $(W, \wedge, \vee, 0, 1, {}^{\smile})$, by a *filter* (resp. *ideal*) of W we mean a filter (resp. ideal) of the underlying lattice $(W, \wedge, \vee, 0, 1)$.

For any $A \subseteq W$, by $A^{\smile}$ we will denote the set:

$$A^{\smile} = \{a^{\smile} \in W : a \in A\}. \tag{10}$$

We have the following:

Lemma 8. *Let $(W, \wedge, \vee, 0, 1, {}^{\smile})$ be an LC algebra. Then the following assertions hold for all $A, B \subseteq W$:*

- **(i)** $A^{\smile} = \{a \in W : a^{\smile} \in A\}$
- **(ii)** $A^{\smile\smile} = A$
- **(iii)** $A \subseteq B$ *iff* $A^{\smile} \subseteq B^{\smile}$
- **(iv)** $(-A)^{\smile} = -(A^{\smile})$
- **(v)** $(A \cup B)^{\smile} = A^{\smile} \cup B^{\smile}$
- **(vi)** $(A \cap B)^{\smile} = A^{\smile} \cap B^{\smile}$.

Proof. By way of example we show **(ii)** and **(iii)**.

(ii) Let $a \in W$. Then $a \in A^{\smile\smile}$ iff $a^{\smile} \in A^{\smile}$ iff $a^{\smile\smile} \in A$ iff $a \in A$.

(iii) **($\Rightarrow$)** Let $A, B \subseteq W$ be such that $A \subseteq B$ and let $a \in A^{\smile}$. Hence, by definition (10), $a^{\smile} \in A$, which by assumption implies $a^{\smile} \in B$.

($\Leftarrow$) Assume that $A^{\smile} \subseteq B^{\smile}$ and $a \in A$. By **(C.1)**, $a^{\smile\smile} \in A$, so $a^{\smile} \in A^{\smile}$, which by assumption gives $a^{\smile} \in B^{\smile}$. It follows that $a^{\smile\smile} \in B$, and hence, $a \in B$. ■

Lemma 9. *Let $(W, \wedge, \vee, 0, 1, {}^{\smile})$ be an LC algebra and let $A \subseteq W$. Then the following assertions hold:*

- **(i)** *If A is a filter of W, then so is $A^{\smile}$*
- **(ii)** *If A is an ideal of W, then so is $A^{\smile}$.*

Proof.

(i) Let A be a filter of W and $a, b \in W$ such that $a \leqslant b$ and $a \in A^{\smile}$. Then $a^{\smile} \in A$, and, by Lemma 7**(ii)** we also have $a^{\smile} \leqslant b^{\smile}$. This implies, $b^{\smile} \in A$, and thus, $b \in A^{\smile}$.
Let $a, b \in A^{\smile}$. This means that $a^{\smile} \in A$ and $b^{\smile} \in A$, so $a^{\smile} \wedge b^{\smile} \in A$ since A

is a filter. By Lemma 7**(iii)**, $a^{\smile} \wedge b^{\smile} = (a \wedge b)^{\smile}$, so that $(a \wedge b)^{\smile} \in A = A^{\smile\smile}$ by Lemma 8**(ii)**. Then $(a \wedge b)^{\smile\smile} \in A^{\smile}$, or equivalently, $a \wedge b \in A^{\smile}$.

(ii) Let A be an ideal of W and let $a, b \in W$. Assume that $b \in A^{\smile}$ and $a \leqslant b$. Then $b^{\smile} \in A$ and by Lemma 7**(ii)**, $a^{\smile} \leqslant b^{\smile}$. Hence $a^{\smile} \in A$, so $a \in A^{\smile}$.
Let $a, b \in A^{\smile}$. Then $a^{\smile} \in A$ and $b^{\smile} \in A$. Since A is an ideal, $a^{\smile} \vee b^{\smile} \in A$. By axiom **(C.1)**, $a^{\smile} \vee b^{\smile} = (a \vee b)^{\smile}$. Hence $(a \vee b)^{\smile} \in A$, so $a \vee b \in A^{\smile}$. ■

4.1 LC frames

Definition 9. An ***LC frame*** is a relational system $(X, \leqslant_1, \leqslant_2, C)$ such that $(X, \leqslant_1, \leqslant_2)$ is a doubly ordered set and C is a mapping $C : X \to X$ satisfying the following conditions for all $x, y \in X$:

(MC.1) $x \leqslant_1 y$ implies $C(x) \leqslant_1 C(y)$
(MC.2) $x \leqslant_2 y$ implies $C(x) \leqslant_2 C(y)$
(SC) $C(C(x)) = x$. □

Given an LC frame $(X, \leqslant_1, \leqslant_2, C)$ let us define a mapping $^{\curlyvee} : 2^X \to 2^X$ as follows: for every $A \subseteq X$,

$$A^{\curlyvee} = \{C(x) : x \in A\}. \tag{11}$$

The following two lemmas present some properties of $^{\curlyvee}$.

Lemma 10. *Let $(X, \leqslant_1, \leqslant_2, C)$ be an LC frame and let $^{\curlyvee}$ be defined by (11). Then for all $A, B \subseteq X$,*

(i) $A^{\curlyvee} = \{x \in X : C(x) \in A\}$
(ii) $A^{\curlyvee\curlyvee} = A$
(iii) $A \subseteq B$ *implies* $A^{\curlyvee} \subseteq B^{\curlyvee}$
(iv) $(A \cap B)^{\curlyvee} = A^{\curlyvee} \cap B^{\curlyvee}$.

Proof. By way of example we show **(ii)** and **(iv)**.

(ii) Let $x \in W$. By **(SC)** and the definition (11) we have the following equivalences: $x \in A$ iff $C(C(x)) \in A$ iff $C(x) \in A^{\curlyvee}$ iff $x \in A^{\curlyvee\curlyvee}$.

(iv)($\subseteq$) Since $A \cap B \subseteq A$, by **(iii)** it follows that $(A \cap B)^{\curlyvee} \subseteq A^{\curlyvee}$. Similarly, $(A \cap B)^{\curlyvee} \subseteq B^{\curlyvee}$. Then $(A \cap B)^{\curlyvee} \subseteq A^{\curlyvee} \cap B^{\curlyvee}$.

($\supseteq$) Since $A^{\curlyvee} \cap B^{\curlyvee} \subseteq A^{\curlyvee}$, from **(iii)** and **(ii)** it follows $(A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee} \subseteq A^{\curlyvee\curlyvee} = A$. Also, $(A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee} \subseteq B$. Then $(A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee} \subseteq A \cap B$, so again by **(ii)** and **(iii)**, $A^{\curlyvee} \cap B^{\curlyvee} = (A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee\curlyvee} \subseteq (A \cap B)^{\curlyvee}$. ■

Lemma 11. *Let $(X, \leqslant_1, \leqslant_2, C)$ be an LC frame and let $^{\curlyvee}$ be defined by (11). Then for all $A, B \subseteq X$,*

(i) $l(A^{\curlyvee}) = l(A)^{\curlyvee}$
(ii) $r(A^{\curlyvee}) = r(A)^{\curlyvee}$.
(iii) *if A is l–stable, then so is $A^{\curlyvee}$.*

Proof. By way of example we show **(i)** and **(iii)**.

(i) **($\subseteq$)** Let $x \notin l(A)^{\curlyvee}$. By the definition (11), this means that $C(x) \notin l(A)$, so there exists $y \in X$ such that **(i.1)** $C(x) \leqslant_1 y$, and **(i.2)** $y \in A$. By **(MC.1)**, **(i.1)** implies $C(C(x)) \leqslant_1 C(y)$, so by **(SC)** we get **(i.3)** $x \leqslant_1 C(y)$. Next, **(i.2)** and **(SC)** imply $C(C(y)) \in A$, whence $C(y) \in A^{\curlyvee}$, which together with **(i.3)** implies $x \notin l(A^{\curlyvee})$.
($\supseteq$) can be proved in the similar way.

(iii) Let A be l–stable. By **(i)** and **(ii)**, $l(r(A^{\curlyvee})) = l(r(A)^{\curlyvee}) = l(r(A))^{\curlyvee} = A^{\curlyvee}$, so $A^{\curlyvee}$ is l–stable. ■

4.2 Complex algebras of LC frames

Definition 10. Let $(X, \leqslant_1, \leqslant_2, C)$ be an LC frame. By the ***complex algebra of*** X we mean a structure $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, {}^{\curlyvee})$ with the operations defined by (3), (4), (11) and the constants defined by (5) and (6). □

Theorem 3. *The complex algebra of an LC frame is an LC algebra.*

Proof. From Lemma 10**(ii)**, $A^{\curlyvee\curlyvee} = A$, so it suffices to show that $(A \sqcup B)^{\curlyvee} = A^{\curlyvee} \sqcup B^{\curlyvee}$. For every $x \in X$,

$x \in (A \sqcup B)^{\curlyvee}$	iff	$C(x) \in A \sqcup B$	by the definition of ${}^{\curlyvee}$
	iff	$C(x) \in l(r(A) \cap r(B))$	by the definition of $\sqcup$
	iff	$x \in l(r(A) \cap r(B))^{\curlyvee}$	by the definition of ${}^{\curlyvee}$
	iff	$x \in l((r(A) \cap r(B))^{\curlyvee})$	by Lemma 11**(i)**
	iff	$x \in l(r(A)^{\curlyvee} \cap r(B)^{\curlyvee})$	by Lemma 10**(iv)**
	iff	$x \in l(r(A^{\curlyvee}) \cap r(B^{\curlyvee}))$	by Lemma 11**(ii)**
	iff	$x \in A^{\curlyvee} \sqcup B^{\curlyvee}$. ■	

4.3 Canonical frames of LC algebras

Let $(W, \wedge, \vee, 0, 1, {}^{\smile})$ be an LC algebra. As usual, $FIP(W)$ and $X(W)$ denote the family of all filter–ideal pairs (resp. maximal filter–ideal pairs) of W.

First, observe the following:

Lemma 12. $(x_1, x_2) \in FIP(W)$ *iff* $(x_1^{\smile}, x_2^{\smile}) \in FIP(W)$.

Proof. **(⇒)** Let $(x_1, x_2) \in FIP(W)$. From Lemma 9 it follows that $x_1^{\smile}$ is a filter of W and $x_2^{\smile}$ is an ideal of W. Note that $\emptyset^{\smile} = \emptyset$. Then, by Lemma 8**(vi)** we get that $x_1^{\smile} \cap x_2^{\smile} = \emptyset$, so $(x_1^{\smile}, x_2^{\smile}) \in FIP(W)$

(⇐) Let $x_1, x_2 \subseteq W$ be such that $(x_1^{\smile}, x_2^{\smile}) \in FIP(W)$. Then, by Lemmas 8**(ii)** and 9, $x_1 = x_1^{\smile\smile}$ is a filter of W and $x_2 = x_2^{\smile\smile}$ is an ideal of W. Next, from Lemma 8**(vi)**, $(x_1 \cap x_2)^{\smile} = \emptyset$, so $x_1 \cap x_2 = (x_1 \cap x_2)^{\smile\smile} = \emptyset$. Whence $(x_1, x_2) \in FIP(W)$. ■

Let us now define a mapping $C^{\star} : FIP(W) \to FIP(W)$ as follows: for every $x \in FIP(W)$,

$$C^{\star}(x) = (x_1^{\smile}, x_2^{\smile}). \tag{12}$$

Lemma 13. *If x is a maximal filter–ideal pair of W, then so is $C^{\star}(x)$.*

Proof. Let $x = (x_1, x_2) \in FIP(W)$. Assume that $(x_1^{\smile}, x_2^{\smile})$ is not maximal. By Proposition 1, it can be extended to the maximal filter–ideal pair, say $y = (y_1, y_2)$. Then $x_1^{\smile} \subseteq y_1$, $x_2^{\smile} \subseteq y_2$ and $(x_1^{\smile}, x_2^{\smile}) \neq (y_1, y_2)$. By Lemma 8**(ii)** and 8**(iii)** we get $x_1 \subseteq y_1^{\smile}$, $x_2 \subseteq y_2^{\smile}$ and $(x_1, x_2) \neq (y_1^{\smile}, y_2^{\smile})$, which means that (x_1, x_2) is not a maximal filter–ideal pair. ■

Definition 11. Let $(W, \wedge, \vee, 0, 1, {}^{\smile})$ be an LC algebra. A ***canonical frame*** of W is a structure $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^{\star})$, where $\preccurlyeq_1$, $\preccurlyeq_2$ and $C^{\star}$ are defined by (7), (8) and (12), respectively. □

Theorem 4. *The canonical frame of an LC algebra is an LC frame.*

Proof. Let $x, y \in X(W)$ and assume that $x \preccurlyeq_1 y$. This means that $x_1 \subseteq y_1$. By Lemma 8**(iii)**, $x_1^{\smile} \subseteq y_1^{\smile}$, so $C^{\star}(x) \preccurlyeq_1 C^{\star}(y)$. Hence **(MC.1)** holds. In the analogous way we can show that **(MC.2)** holds. Finally, let $x = (x_1, x_2) \in X(W)$. Then we have $C^{\star}(C^{\star}(x)) = (x_1^{\smile\smile}, x_2^{\smile\smile}) = (x_1, x_2) = x$ by Lemma 8**(ii)**, so the condition **(SC)** also holds. ■

4.4 Relational representation of LC algebras

In this section we conclude our discussion of LC algebras by showing their relational representability.

Theorem 5 (Representation theorem for LC algebras). *Every LC algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. Let $(W, \wedge, \vee, 0, 1, ^{\smile})$ be an LC algebra, $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^{\star})$ be the canonical frame of W and let $(L(X(W)), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, ^{\curlyvee})$ be the complex algebra of the canonical frame of W. By Theorems 3 and 4 it follows that $L(X(W))$ is an LC algebra, so it suffices to show that W is isomorphic to a subalgebra of $L(X(W))$.
Let the mapping $h : W \to 2^{X(W)}$ be defined as in (9), i.e. $h(a) = \{x \in X(W) : a \in x_1\}$, $a \in W$. We show that for every $a \in W$, $h(a^{\smile}) = h(a)^{\curlyvee}$. For every $x \in X(W)$ and for every $a \in W$ we have: $x \in h(a^{\smile})$ iff $a^{\smile} \in x_1$ iff $a \in x_1^{\smile}$ iff $C^{\star}(x) \in h(a)$ iff $x \in h(a)^{\curlyvee}$.
By Theorem 1 h preserves the lattice operations and is injective. ■

5 LP algebras

LP algebras are a join of a not necessary distributive bounded lattice with a monoid. The monoid product operation is an abstract counterpart to the relational composition and the unit element of the monoid corresponds to the identity relation.

Definition 12. An ***LP algebra*** is a structure $(W, \wedge, \vee, 0, 1, \odot, 1')$ such that $(W, \wedge, \vee, 0, 1)$ is a bounded lattice and for all $a, b, c \in W$,

(P.1) $a \odot 1' = 1' \odot a = a$
(P.2) $a \odot (b \odot c) = (a \odot b) \odot c$
(P.3) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$
(P.4) $(a \vee b) \odot c = (a \odot c) \vee (b \odot c)$. □

Lemma 14. *Let $(W, \wedge, \vee, 0, 1, \odot, 1')$ be an LP algebra. For all $a, b, c \in W$, if $a \leqslant b$, then*

(i) $c \odot a \leqslant c \odot b$
(ii) $a \odot c \leqslant b \odot c$.

Proof.
(i) Let $a \leqslant b$. Then $a \vee b = b$, so $c \odot (a \vee b) = c \odot b$. Hence, by axiom **(P.3)**, we get $(c \odot a) \vee (c \odot b) = c \odot b$, which implies $c \odot a \leqslant c \odot b$.
(ii) This can be proved in an analogous way. ■

5.1 LP frames

In this section we follow the developments of Allwein and Dunn ([1]). However, the difference is that here we consider an abstract notion of an LP frame, not only the canonical frame of an LP algebra.

Definition 13. An ***LP frame*** is a relational system $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ such that $(X, \leqslant_1, \leqslant_2)$ is a doubly ordered set, R, S, Q are ternary relations on X and $I \subseteq X$ is a unary relation on X such that the following conditions are satisfied: for all $x, x', y, y', z, z' \in X$,

A. Monotonicity conditions

(MP.1) $R(x,y,z) \;\&\; x' \leqslant_1 x \;\&\; y' \leqslant_1 y \;\&\; z \leqslant_1 z' \Rightarrow R(x', y', z')$

(MP.2) $S(x,y,z) \;\&\; x \leqslant_2 x' \;\&\; y' \leqslant_1 y \;\&\; z' \leqslant_2 z \Rightarrow S(x', y', z')$

(MP.3) $Q(x,y,z) \;\&\; x' \leqslant_1 x \;\&\; y \leqslant_2 y' \;\&\; z' \leqslant_2 z \Rightarrow Q(x', y', z')$

(MP.4) $I(x) \;\&\; x \leqslant_1 x' \Rightarrow I(x')$

B. Stability conditions

(SP.1) $R(x,y,z) \Rightarrow \exists x'' \in X \; (x \leqslant_1 x'' \;\&\; S(x'', y, z))$

(SP.2) $R(x,y,z) \Rightarrow \exists y'' \in X \; (y \leqslant_1 y'' \;\&\; Q(x, y'', z))$

(SP.3) $S(x,y,z) \Rightarrow \exists z'' \in X \; (z \leqslant_2 z'' \;\&\; R(x, y, z''))$

(SP.4) $Q(x,y,z) \Rightarrow \exists z'' \in X \; (z \leqslant_2 z'' \;\&\; R(x, y, z''))$

(SP.5) $\exists u \in X(R(x,y,u) \;\&\; Q(x',u,z)) \Rightarrow \exists w \in X(R(x',x,w) \;\&\; S(w,y,z))$

(SP.6) $\exists u \in X(R(x,y,u) \;\&\; S(u,z,z')) \Rightarrow \exists w \in X(R(y,z,w) \;\&\; Q(x,w,z'))$

(SP.7) $I(x) \;\&\; (R(x,y,z) \text{ or } R(y,x,z)) \Rightarrow y \leqslant_1 z$

(SP.8) $\exists u \in X(I(u) \;\&\; S(u,x,x))$

(SP.9) $\exists u \in X(I(u) \;\&\; Q(x,u,x))$. $\square$

Lemma 15. *For every LP frame $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ and every l–stable subset $A \subseteq X$ it holds for all $x, y, z \in X$:*

$$I(x) \;\&\; (y \in A) \;\&\; (R(x,y,z) \text{ or } R(y,x,z)) \Rightarrow z \in A.$$

Proof.

Let $x, y, z \in X$ and assume that **(1)** $I(x)$ **(2)** $y \in A$ and **(3)** $R(x,y,z)$ or $R(y,x,z)$. By **(SP.7)**, we get from **(1)** and **(3)** that **(4)** $y \leqslant_1 z$. Since A is l–stable, by Lemma 1**(i)** it is $\leqslant_1$–increasing, so **(2)** and **(4)** imply $z \in A$. ∎

For an LP frame $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$, let us define two mappings $\odot_Q$, $\odot_S$: $2^X \times 2^X \to 2^X$ by: for all $A, B \subseteq X$,

$$A \odot_Q B = \{z \in X : \forall x, y \in X \; (Q(x,y,z) \;\&\; x \in A \Rightarrow y \in r(B))\} \tag{13}$$

$$A \odot_S B = \{z \in X : \forall x, y \in X \; (S(x,y,z) \;\&\; y \in B \Rightarrow x \in r(A)\}. \tag{14}$$

Lemma 16. *For any $A, B \subseteq X$,*

(i) *$A \odot_Q B$ and $A \odot_S B$ are $\leqslant_2$–increasing*

(ii) *if A and B are l–stable, then $A \odot_Q B$ and $A \odot_S B$ are r–stable.*

Proof.

(i) Let $A, B \subseteq X$ and suppose that $A \odot_Q B$ is not $\leqslant_2$–increasing. Then, by Definition 2, there exist $x, y \in X$ such that **(i.1)** $x \in A \odot_Q B$, **(i.2)** $x \leqslant_1 y$ and **(i.3)** $y \notin A \odot_Q B$. By the definition (13), **(i.3)** means that there exist $u, w \in X$ such that **(i.4)** $Q(u, w, y)$, **(i.5)** $u \in A$ and **(i.6)** $w \notin r(B)$. However, by **(MP.3)**, **(i.2)** and **(i.4)** imply $Q(u, w, x)$, which together with **(i.5)** and **(i.6)** gives $x \notin A \odot_Q B$ – a contradiction with **(i.1)**.
In the similar way one can show that $A \odot_S B$ is $\leqslant_2$–increasing.

(ii) Let $A, B \subseteq X$ be l–stable sets. By **(i)**, $A \odot_Q B$ is $\leqslant_2$–increasing. Therefore, by Lemma 4**(iii)**, $A \odot_Q B \subseteq r(l(A \odot_Q B))$, so it suffices to show that $r(l(A \odot_Q B)) \subseteq A \odot_Q B$.
Let $z \in X$ and assume that $z \notin A \odot_Q B$. We show that **(ii.1)** $z \notin r(l(A \odot_Q B))$. By the definition (13) of $\odot_Q$, there exist $x, y \in X$ such that **(ii.2)** $Q(x, y, z)$, **(ii.3)** $x \in A$ and **(ii.4)** $y \notin r(B)$. From **(ii.4)**, there exists $y' \in X$ such that **(ii.5)** $y \leqslant_2 y'$ and **(ii.6)** $y' \in B$. By **(MP.3)**, **(ii.2)** and **(ii.5)** imply $Q(x, y', z)$, which by **(SP.4)** gives that there exists $z' \in X$ such that **(ii.7)** $z \leqslant_2 z'$ and **(ii.8)** $R(x, y', z')$.
Now we show that **(ii.9)** $z' \in l(A \odot_Q B)$. Let $z'' \in X$ and assume that **(ii.10)** $z' \leqslant_1 z''$. Hence, by **(MP.1)**, from **(ii.8)** we get that $R(x, y', z'')$, which by **(SP.2)** implies that there exists $y'' \in X$ such that **(ii.11)** $y' \leqslant_1 y''$ and **(ii.12)** $Q(x, y'', z'')$. Next, **(ii.10)** and **(ii.12)** imply **(ii.13)** $Q(x, y'', z')$ by **(MP.3)**. Furthermore, since B is l–stable, it is $\leqslant_1$–increasing, so from **(ii.6)** and **(ii.11)**, $y'' \in B$. It follows that **(ii.14)** $y'' \notin r(B)$. Then we have that for some $x, y'' \in X$, **(ii.3)**, **(ii.13)** and **(ii.14)** hold, which means that $z' \notin A \odot_Q B$. Since z'' was an arbitrary element satisfying **(ii.10)**, we obtain that for any $z'' \in X$, $z' \leqslant_1 z''$ implies $z' \notin A \odot_Q B$, so **(ii.9)** follows. Finally, from **(ii.7)** and **(ii.9)** we obtain $z \notin rl(A \odot_Q B)$.
Proceeding in the similar way we can show that $A \odot_S B$ is r–stable. ∎

Lemma 17. *For every LP frame $(X, \leqslant_1, \leqslant_2, R, S, C, I)$ and for all l–stable subsets $A, B \subseteq X$, $A \odot_S B = A \odot_Q B$.*

Proof. ($\subseteq$) Let $z \in X$ and assume that $z \notin A \odot_Q B$. We show that $z \notin A \odot_S B$. By assumption, there exist $x, y \in X$ such that **(1)** $Q(x, y, z)$, **(2)** $x \in A$ and **(3)** $y \notin r(B)$. From **(3)**, there is $y' \in X$ such that **(4)** $y \leqslant_2 y'$ and **(5)** $y' \in B$. Next, from **(1)** and **(4)** we get by **(MP.3)** that $Q(x, y', z)$, which by **(SP.4)** implies that there exists $z' \in X$ such that **(7)** $z \leqslant_2 z'$ and **(8)** $R(x, y', z')$. Furthermore, applying **(SP.1)**, we get from **(8)** that there exists $x' \in X$ such that **(9)** $x \leqslant_1 x'$ and **(10)** $S(x', y', z')$. Also, by **(MP.2)**, from **(7)** and **(10)** it follows that **(11)** $S(x', y', z)$. From **(2)**, **(9)** and the assumption that A is l–stable we get that $x' \in A$. Then $x \notin r(A)$, which together with **(5)** and **(10)** implies **(12)** $z' \notin A \odot_S B$. By Lemma 16**(i)**, $A \odot_S B$ is $\leqslant_2$–increasing. Hence, by **(7)** and **(12)** we get $z \notin A \odot_S B$.

The proof of ($\supseteq$) is similar. ∎

Let us define a mapping $\otimes : 2^X \times 2^X \to 2^X$ as follows: for all $A, B \subseteq X$,

$$A \otimes B = l(A \odot_Q B). \tag{15}$$

Lemma 18. *Let $A, B \subseteq X$. If A and B are l–stable sets, then so is $A \otimes B$.*

Proof. By Lemma 16**(ii)**, $A \odot_Q B$ is r–stable, so from Lemma 1**(vi)** we get that $l(A \odot_Q B)$ is l–stable. ■

Given an LP frame $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$, let us define

$$\mathbf{1'} = l(r(I)) \tag{16}$$

Lemma 19. $\mathbf{1'}$ *is l–stable.*

Proof. Follows from Lemma 1**(ii)** and **(iv)**. ■

5.2 Complex algebras of LP frames

Definition 14. Let $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ be an LP frame. A ***complex algebra of*** X is a structure $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ with operations defined by (3)–(6), (15) and (16). □

Our aim now is to show that complex algebras of LP frames are LP algebras. To this end, we show that any complex algebra of an LP frame satisfies axioms **(P.1)**–**(P.4)**.

First, we prove that axiom **(P.1)** is satisfied in $L(X)$.

Lemma 20. *Let $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ be a complex algebra of an LP frame X. Then for every $A \in L(X)$,*

(i) $\mathbf{1'} \otimes A = A$

(ii) $A \otimes \mathbf{1'} = A$.

Proof.

(i) Let $A \in L(X)$. We show that $\mathbf{1'} \odot_Q A = r(A)$. Then $l(\mathbf{1'} \odot_Q A) = l(r(A))$, which by the assumption and the definition (15) gives the result.

($\subseteq$) Let $z \in X$ and assume that $z \notin r(A)$. Then there exists $w \in X$ such that **(i.1)** $z \leqslant_2 w$ and **(i.2)** $w \in A$. From **(SP.8)**, there exists $x \in X$ such that **(i.3)** $I(x)$ and **(i.4)** $S(x, w, w)$. Next, from **(MP.4)**, I is $\leqslant_1$–increasing, so by Lemma 4**(ii)**, $I \subseteq l(r(I))$, which together with **(i.3)** implies $x \in l(r(I))$, or equivalently, by the definition (16), **(i.5)** $x \in \mathbf{1'}$. Also, by **(MP.2)**, **(i.1)** and **(i.4)** imply $S(x, w, z)$. Then, in view of **(SP.3)**, there exists $z' \in X$ such that **(i.6)** $z \leqslant_2 z'$ and **(i.7)** $R(x, w, z')$. Furthermore, **(i.7)** implies by **(SP.2)** that there exists $y \in X$ such that **(i.8)** $w \leqslant_1 y$ and **(i.9)** $Q(x, y, z')$. From **(i.6)** and **(i.9)** we get by **(MP.3)** that **(i.10)** $Q(x, y, z)$. Since A is l–stable, by Lemma 1**(i)** it is $\leqslant_1$–increasing. Then **(i.2)** and **(i.8)** imply $y \in A$, so $y \notin r(A)$, which together with **(i.5)** and **(i.10)** gives $z \notin \mathbf{1'} \odot_Q A$ by the definition (13).

($\supseteq$) Let $z \in X$ and assume that $z \notin \mathbf{1'} \odot_Q A$. We show that $z \notin r(A)$.
By assumption, for some $x, y \in X$ we have: **(i.11)** $Q(x, y, z)$, **(i.12)** $x \in \mathbf{1'}$ and **(i.13)** $y \notin r(A)$. From **(i.13)**, there exists $y' \in X$ such that **(i.14)** $y \leqslant_2 y'$ and **(i.15)** $y' \in A$. By **(MP.3)**, from **(i.11)** and **(i.14)** it follows that $Q(x, y', z)$, which by **(SP.4)** implies that there exists $z' \in X$ such that **(i.16)** $z \leqslant_2 z'$ and **(i.17)** $R(x, y', z')$. It suffices to show that $z' \in A$. Then, by **(i.16)** and the definition (1), $z \notin r(A)$.
Suppose that $z' \notin A$. By l–stability of A, this means that $z' \notin l(r(A))$, i.e. there exists $z'' \in X$ such that **(i.18)** $z' \leqslant_1 z''$ and **(i.19)** $z'' \in r(A)$. Applying **(MP.1)**, from **(i.17)** and **(i.18)**, $R(x, y', z'')$, which by **(SP.1)** implies that there exists $x' \in X$ such that **(i.20)** $x \leqslant_1 x'$ and **(i.21)** $S(x', y', z'')$. From **(i.12)**, $x \in \mathbf{1'} = l(r(I))$. Hence, by **(i.20)** and the definition (2), $x' \notin r(I)$, so there exists $x'' \in X$ such that **(i.22)** $x' \leqslant_2 x''$ and **(i.23)** $x'' \in I$. By **(MP.2)**, **(i.21)** and **(i.22)** imply $S(x'', y', z'')$, which by **(SP.3)** gives that there exists $w \in X$ such that **(i.24)** $z'' \leqslant_2 w$ and **(i.25)** $R(x'', y', w)$. Now, applying Lemma 15, **(i.25)**, **(i.23)** and **(i.17)** imply $w \in A$, which together with **(i.24)** gives $z'' \notin r(A)$, which contradicts **(i.20)**.

(ii) Let $A \in L(X)$. As before, we will show that $A \odot_Q \mathbf{1'} = r(A)$. Hence we will have $A \otimes \mathbf{1'} = l(A \odot_Q \mathbf{1'}) = r(l(A)) = A$.

($\subseteq$) Let $z \in X$ and assume that $z \notin r(A)$. By the definition (2) this means that there exists $x \in X$ such that **(ii.1)** $z \leqslant_2 x$ and **(ii.2)** $x \in A$. From **(SP.9)**, there exists $y \in X$ such that **(ii.3)** $y \in I$ and **(ii.4)** $Q(x, y, x)$. From **(MP.4)**, I is $\leqslant_1$–increasing, so by Lemma 4**(ii)**, $I \subseteq l(r(I)) = \mathbf{1'}$. Then from **(ii.3)** it follows that $y \in \mathbf{1'}$, whence **(ii.5)** $y \notin r(\mathbf{1'})$. Next, by **(MP.3)**, **(ii.1)** and **(ii.4)** imply **(ii.6)** $Q(x, y, z)$. Therefore, there exist $x, y \in X$ such that **(ii.2)**, **(ii.5)** and **(ii.6)** hold, which by the definition (13) means that $z \notin A \odot_Q \mathbf{1'}$.

($\supseteq$) We have to show that $r(A) \subseteq A \odot_Q \mathbf{1'}$. Assume that $z \notin A \odot_Q \mathbf{1'}$. By the definition (13), there exist $x, y \in X$ such that **(ii.7)** $Q(x, y, z)$, **(ii.8)** $x \in A$ and **(ii.9)** $y \notin r(\mathbf{1'})$. From **(ii.9)**, there exists $y' \in X$ such that **(ii.10)** $y \leqslant_2 y'$ and **(ii.11)** $y' \in \mathbf{1'}$. By **(MP.3)**, **(ii.7)** and **(ii.10)** imply $Q(x, y', z)$, from which, by **(SP.4)**, it follows that there exists $z' \in X$ such that **(ii.12)** $z \leqslant_2 z'$ and

$R(x, y', z')$. Proceeding in the similar way as in the proof of **($\supseteq$)** in **(i)**, we can show that $z' \in A$, which together with **(ii.12)** implies $z \notin r(A)$. ■

The following lemma states that **(P.2)** is satisfied in $L(X)$.

Lemma 21. *Let* $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ *be the complex algebra of an LP frame* X*. Then for all* $A, B, C \in L(X)$*, the following equality holds:*

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C.$$

Proof. Let $A, B, C \subseteq X$ be l–stable sets. In view of Lemma 17 it suffices to show that $A \odot_Q (B \otimes C) = (A \otimes B) \odot_S C$.

($\subseteq$) Let $z \in X$ and assume that $z \notin (A \otimes B) \odot_S C$. By the definition (14), this means that there exist $x, y \in X$ such that **(1)** $S(x, y, z)$, **(2)** $y \in C$ and **(3)** $x \notin r(A \otimes B)$. But $r(A \otimes B) = r(l(A \odot_Q B))$ by the definition (15). Next, by Lemma 17, $r(l(A \odot_Q B)) = r(l(A \odot_S B))$. Moreover, $A \odot_S B$ is r–stable by Lemma 16**(ii)**, so $r(l(A \odot_S B)) = A \odot_S B$. Then we have $r(A \otimes B) = A \odot_S B$. Hence, from **(3)** we get $x \notin A \odot_S B$, which means that there exist $u, v \in X$ such that **(4)** $S(u, v, x)$, **(5)** $v \in B$ and **(6)** $u \notin r(A)$. From **(6)**, there exists $u' \in X$ such that **(7)** $u \leqslant_2 u'$ and **(8)** $u' \in A$. By the monotonicity condition **(MP.2)**, **(4)** and **(7)** imply $S(u', v, x)$, which by **(SP.3)** gives that there exists $x' \in X$ such that **(9)** $x \leqslant_2 x'$ and **(10)** $R(u', v, x')$. Applying again **(MP.2)**, from **(1)** and **(9)** we get that $S(x', y, z)$. Therefore, there exists $x' \in X$ such that $R(u'v, x')$ and $S(x', y, z)$. By **(SP.6)**, this implies that there exists $w \in X$ such that **(11)** $R(v, y, w)$ and **(12)** $Q(u', w, z)$. Next, by **(SP.1)**, **(11)** implies that there exists $v' \in X$ such that **(13)** $v \leqslant_1 v'$ and **(14)** $S(v', y, w)$. Since B is l–stable, it is $\leqslant_1$–increasing. Then **(5)** and **(13)** imply $v' \in B$, so **(15)** $v' \notin r(B)$. We have then obtained that there exist $y, v' \in X$ such that **(2)**, **(14)** and **(15)** hold, which by the definition (14) means that **(16)** $w \notin B \odot_S C$. From Lemma 16, $B \odot_S C$ is r–stable, so $B \odot_S C = r(l(B \odot_S C)) = r(l(B \odot_Q C)) = r(B \otimes C)$ by Lemma 17. Whence **(16)** implies **(17)** $w \notin r(B \otimes C)$. Then we finally get that there exist $u', w \in X$ such that **(8)**, **(12)** and **(17)** hold. By the definition (13), this means that $z \notin A \odot_Q (B \otimes C)$.

Proceeding in the similar way, and using **(SP.5)**, **($\supseteq$)** can be proved. ■

Finally, we show that axioms **(P.3)** and **(P.4)** are satisfied in complex algebras of LP frames.

Lemma 22. *Let* $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ *be the complex algebra of an LP frame* X*. Then for all* $A, B, C \in L(X)$*,*

- **(i)** $A \otimes (B \sqcup C) = (A \otimes B) \sqcup (A \otimes C)$
- **(ii)** $(B \sqcup C) \otimes A = (B \otimes A) \sqcup (C \otimes A)$.

Proof.

(i) Let $A,B,C \subseteq X$ be l–stable sets. First we show that **(i.1)** $A \odot_Q (B \sqcup C) = (A \odot_Q B) \cap (A \odot_Q C)$.
For every $x \in X$ the following equivalences hold:

$$\begin{array}{ll} x \in (A \odot_Q B) \cap (A \odot_Q C) & \\ \quad \text{iff} & x \in A \odot_Q B \ \&\ x \in A \odot_Q C \\ \quad \text{iff} & [\forall y,z \in X\ (Q(y,z,x)\ \&\ y \in A \Rightarrow z \in r(B))] \\ & \quad \&\ [\forall y,z \in X\ (Q(y,z,x)\ \&\ y \in A \Rightarrow z \in r(C))] \\ \quad \text{iff} & \forall y,z \in X\ (Q(y,z,x)\ \&\ y \in A \Rightarrow z \in r(B) \cap r(C)). \end{array}$$

By Lemma 1**(vii)**, $r(B) \cap r(C)$ is r–stable, so $r(B) \cap r(C) = r(l(r(B) \cap r(C)))$. Hence, by the definition (4) we get

$$\begin{array}{ll} x \in (A \odot_Q B) \cap (A \odot_Q C) & \\ \quad \text{iff} & \forall y,z, \in X\ (Q(y,z,x)\ \&\ y \in A \Rightarrow z \in r(l(r(B) \cap r(C)))) \\ \quad \text{iff} & x \in A \odot_Q l(r(B) \cap r(C)) \\ \quad \text{iff} & x \in A \odot_Q (B \sqcup C). \end{array}$$

So **(i.1)** holds. Hence **(i.2)** $l(A \odot_Q (B \sqcup C)) = l((A \odot_Q B) \cap (A \odot_Q C))$. Note that **(i.3)** $l(A \odot_Q (B \sqcup C)) = A \otimes (B \sqcup C)$. Next, by Lemma 16**(ii)**, $A \odot_Q B$ is r–stable, so $r(l(A \odot_Q B)) = A \odot_Q B$. Using again the definition (4), we get

$$\begin{array}{rcl} l((A \odot_Q B) \cap (A \odot_Q C)) & = & l(r(l(A \odot_Q B)) \cap r(l(A \odot_Q C))) \\ & = & l(A \odot_Q B) \sqcup l(A \odot_Q C) \\ & = & (A \otimes B) \sqcup (A \otimes C). \end{array}$$

Hence, by **(i.2)** and **(i.3)** we get the required result.

(ii) can be proved in the similar way. ∎

From Lemmas 20, 21 and 22 we get:

Theorem 6. *The complex algebra of an LP frame is an LP algebra.* ∎

5.3 Canonical frames of LP algebras

Let $(W, \wedge, \vee, 0, 1, \odot, 1')$ be an LP algebra. As before, by a *filter* (resp. *ideal*) of W we mean a filter (resp. ideal) of the underlying lattice $(W, \wedge, \vee, 0, 1)$. We will write $X(W)$ to denote the family of all maximal filter–ideal pairs of the lattice reduct of W.

Let us define the following ternary relations on $X(W)$: for all $x, y, z \in X(W)$,

$$R^\star(x,y,z) \quad \text{iff} \quad (\forall a, b \in W)\ a \in x_1\ \&\ b \in y_1 \Rightarrow a \odot b \in z_1 \tag{17}$$

$$S^\star(x,y,z) \quad \text{iff} \quad (\forall a, b \in W)\ a \odot b \in z_2\ \&\ b \in y_1 \Rightarrow a \in x_2 \tag{18}$$

$$Q^\star(x,y,z) \quad \text{iff} \quad (\forall a, b \in W)\ a \odot b \in z_2\ \&\ a \in x_1 \Rightarrow b \in y_2 \tag{19}$$

Moreover, let

$$I^\star = \{x \in X(W) : 1' \in x_1\}. \tag{20}$$

Definition 15. Let an LP algebra $(W, \wedge, \vee, 0, 1, \odot, 1')$ be given. The structure $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ is called a ***canonical frame of*** W. □

The following two lemmas can be proved as in [1].

Lemma 23. *Let $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ be the canonical frame of an LP algebra. Then $R^\star$, $S^\star$, $Q^\star$ and $I^\star$ satisfy monotonicity conditions* **(MP.1)**–**(MP.4)** *of Definition 13.* ■

Lemma 24. *Let $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ be the canonical frame of an LP algebra. Then $R^\star$, $S^\star$, $Q^\star$ and $I^\star$ satisfy stability conditions* **(SP.1)**–**(SP.9)** *of Definition 13.* ■

Lemmas 23 and 24 imply the following theorem:

Theorem 7. *The canonical frame of an LP algebra is an LP frame.* ■

5.4 Relational representation for LP algebras

Let $(W, \wedge, \vee, 0, 1, \odot, 1')$ be an LP algebra, $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ be its canonical frame and let $(L(X(W)), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ be the complex algebra of $X(W)$. Let the mapping $h : W \to 2^{X(W)}$ be defined as in (9), i.e. for every $a \in W$,

$$h(a) = \{x \in X(W) : a \in x_1\}.$$

Our aim is to show that W is isomorphic to a subalgebra of $L(X(W))$.

To begin, we introduce the following auxiliary notation. Given an LP algebra, for any $A, B \subseteq W$ denote

$$A \odot B = \{a \odot b : a \in A\ \&\ b \in B\}.$$

First, note the following:

Lemma 25. *Let an LP algebra $(W, \wedge, \vee, 0, 1, \odot, 1')$ be given and let F and I be a filter and an ideal of W, respectively. Then*

$$U = \{a \in W : (\{a\} \odot F) \cap I \neq \emptyset\}$$
$$V = \{a \in W : (F \odot \{a\}) \cap I \neq \emptyset\}$$

are ideals of W.

Proof. We show that U is an ideal of W. Let **(1)** $a \in U$ and **(2)** $b \leqslant a$. From the definition of U, **(1)** implies that there is $c \in F$ such that **(3)** $a \odot c \in I$. By Lemma 14**(i)**, **(2)** implies **(4)** $b \odot c \leqslant a \odot c$. Since I is a ideal, **(3)** and **(4)** give **(5)** $b \odot c \in I$. So for some $c \in F$, **(5)** holds, which gives $b \in U$.

Let $a, b \in U$. We shall show that $a \vee b \in U$. By assumption, there exist $c, d \in F$ such that **(6)** $a \odot c \in I$ and **(7)** $b \odot c \in I$. Since $c \wedge d \leqslant c$, by Lemma 14**(i)** we have that $a \odot (c \wedge d) \leqslant a \odot c$ and $b \odot (c \wedge d) \leqslant b \odot c$, so by **(6)** and **(7)** we get $(a \odot (c \wedge d) \in I$ and $(b \odot (c \wedge d) \in I$. Since I is an ideal, $(a \odot (c \wedge d)) \vee (b \odot (c \wedge d)) \in I$. From axiom **(P.4)**, $(a \odot (c \wedge d)) \vee (b \odot (c \wedge d)) = (a \vee b) \odot (c \wedge d)$, so **(8)** $(a \vee b) \odot (c \wedge d) \in I$. Since F is a filter, $c \wedge d \in F$. So we have shown that for some $c' = c \wedge d \in F$, $(a \vee b) \odot c' \in I$, which by the definition of U implies that $a \vee b \in U$.

Proceeding in the similar way we can show that V is an ideal. ∎

Theorem 8 (Representation theorem for LP algebras). *Every LP algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. In view of Theorem 1 it suffices to show that

- **(i)** $h(1') = \mathbf{1'}$
- **(ii)** $h(a \odot b) = h(a) \otimes h(b)$.

(i) Note that by (20), $I^\star = h(1')$, so from Theorem 1**(ii)**, it is an l–stable set. Also, $\mathbf{1'} = l(r(I^\star))$ by (16). Hence $\mathbf{1'} = I^\star$, i.e. $h(1') = \mathbf{1'}$.

(ii) **($\subseteq$)** Let $a, b \in W$, $z \in X(W)$ and assume that $z \in h(a \odot b)$. Then it holds **(ii.1)** $a \odot b \in z_1$. We have to show that $z \in h(a) \otimes h(b)$. By the definition (15) and Lemma 17, this means that for any $w \in X(W)$, if $z \preccurlyeq_1 w$, then $w \notin h(a) \odot_S h(b)$. Assume that $z \preccurlyeq_1 w$, i.e. **(ii.2)** $z_1 \subseteq w_1$. We will show that **(ii.3)** $w \notin h(a) \odot_S h(b)$.

Let $[a)$ be the filter generated by a, i.e. $[a) = \{e \in W : a \leqslant e\}$. Define

$$U = \{c \in W : ([a) \odot \{c\}) \cap w_2 \neq \emptyset\}.$$

By Lemma 25, U is an ideal. We show that **(ii.4)** $b \notin U$. Suppose that $b \in U$. Then there exists $e \in [a)$ such that **(ii.5)** $e \odot b \in w_2$. Since $e \in [a)$, $a \leqslant e$, so

$a \vee e = e$. Hence **(ii.6)** $(a \vee e) \odot b = e \odot b$. By axiom **(P.4)**, $(a \vee e) \odot b = (a \odot b) \vee (e \odot b)$. So we have $(a \odot b) \vee (e \odot b) = e \odot b$, whence $a \odot b \leqslant e \odot b$. Then, since w_2 is an ideal, by **(ii.5)** we get $a \odot b \in w_2$, so $a \odot b \notin w_1$, which by **(ii.2)** gives $a \odot b \notin z_1$ – a contradiction with **(ii.1)**.
Then $([b), U) \in FIP(W)$. Let (y_1, y_2) be its extension to the maximal filter–ideal pair. Hence **(ii.6)** $[b) \subseteq y_1$ and **(ii.7)** $U \subseteq y_2$. From **(ii.6)**, $b \in y_1$, so **(ii.8)** $y \in h(b)$.
Next, let us consider the set

$$V = \{c \in W : (\{c\} \odot y_1) \cap w_2 \neq \emptyset\}.$$

By Lemma 25, V is an ideal. We show that **(ii.9)** $a \notin V$. Suppose that $a \in V$. Then there exists $e \in L$ such that **(ii.10)** $e \in y_1$ and **(ii.11)** $a \odot e \in w_2$. By the definition of U, **(ii.11)** means that $e \in U$, so by **(ii.7)**, $e \in y_2$. Whence $e \notin y_1$, which contradicts **(ii.10)**. So **(ii.9)** was proved. Then $([a), V) \in FIP(W)$. Let (x_1, x_2) be its extension to the maximal filter–ideal pair. Then we have **(ii.12)** $[a) \subseteq x_1$ and **(ii.13)** $V \subseteq x_2$. From **(ii.13)**, $a \in x_1$, so $x \in h(a)$, which implies **(ii.14)** $x \notin r(h(a))$.
Finally, we show that **(ii.15)** $S^\star(x, y, w)$. By the definition (18) of $S^\star$, this means that for all $c, d \in W$, $c \notin x_2$ and $d \in y_1$ imply $c \odot d \notin w_2$. Let $c, d \in W$ be such that $c \notin x_2$ and $d \in y_1$. It suffices to show that $c \odot d \notin e_2$. Indeed, $c \notin x_2$ implies by **(ii.13)** that $c \notin V$. From the definition of V, this gives that for any $e \in y_1$, $c \odot e \notin w_2$, so in particular $c \odot d \notin w_2$. Therefore, for some $x, y \in X(W)$, **(ii.8)**, **(ii.14)** and **(ii.15)** hold – by the definition (14) of $\odot_S$ **(ii.3)** follows.

(⊇) Let $a, b \in L$ and let $z \in X(W)$. Assume that $z \in h(a) \otimes h(b)$. By the definition (15) this is equivalent to **(ii.16)** $z \in l(h(a) \odot_Q h(b))$. Let $y \in X(W)$ be such that **(ii.17)** $z_1 \subseteq y_1$. Then from **(ii.16)**, $y \notin h(a) \odot_Q h(b)$, which means that there exist $x, w \in X(W)$ such that **(ii.18)** $Q^\star(x, w, y)$, **(ii.19)** $x \in h(a)$ and **(ii.20)** $w \notin r(h(b))$. From **(ii.19)**, $a \in x_1$. Next, by Theorem 1**(i)**, **(ii.20)** implies **(ii.21)** $b \notin w_2$. By the definition (19), **(ii.18)** gives that for all $a, b \in W$, $a \odot b \in y_2$ and $a \in x_1$ imply $b \in w_2$. Hence, from **(ii.19)** and **(ii.21)**, $a \odot b \notin y_2$. Applying again Theorem 1**(i)**, we obtain $y \notin r(h(a \odot b))$, which together with **(ii.17)** gives $z \in l(r(h(a \odot b)))$. Since by Theorem 1**(ii)**, $h(a \odot b)$ is l–stable, we finally get $z \in h(a \odot b)$. ∎

6 LCP algebras

LCP algebras are meant to be relation algebras based on arbitrary bounded lattices. Their axioms consist of the axioms **(C.1)** and **(C.2)** of converse, the axioms **(P.1)**,…,**(P.4)** of product and an additional axiom which tells us how converse and product are related with each other. In the axiomatization of classical relation algebras, this is done by postulating that converse distributes over composition and also by the axiom

$$a \odot -(a^\smile \odot -b) \leqslant b, \tag{21}$$

where $-x$ is the Boolean complement of x. It is well known that (21) is equivalent to de Morgan's *Theorem K*, one form of which states that

$$a \odot b \leqslant -c \text{ iff } a^{\smile} \odot c \leqslant -b \text{ iff } c \odot b^{\smile} \leqslant -a. \tag{22}$$

In our present setting, we do not have complementation as a distinguished operation, and thus, we cannot use (21). It may be argued that one could use the complement free version of (22), namely,

$$(a \odot b) \wedge c = 0 \text{ iff } (a^{\smile} \odot c) \wedge b = 0 \text{ iff } (c \odot b^{\smile}) \wedge a = 0. \tag{23}$$

However, it is not quite clear whether this is useful because of the following: Suppose that W is an LCP algebra (formally defined below), and let $m \notin W$. Set $W' = W \cup \{m\}$, and extend ordering and the operations of W over W' by

$$m \leqslant x, \quad m^{\smile} = m, \quad m \odot x = x \odot m = m$$

for all $x \in W'$. In other words, we are adding a new smallest element to W.

Lemma 26.

1. *W' is an LCP algebra which satisfies* (23)
2. *If $\tau = \sigma$ is an equation of LCP algebras not containing* 0*, then*

$$W \models \tau = \sigma \textit{ iff } W' \models \tau = \sigma.$$

Proof. The idea is to show that the equational classes generated by W and W' are the same when 0 is omitted in the signature. Suppose that $Eq(W)$ and $Eq(W')$ are these classes. It suffices to show that $W \in Eq(W')$ and $W' \in Eq(W)$. The first claim follows from the fact that W is a subalgebra of W'. The second claim can be seen as follows. Let $f : W' \to W \times W$ be defined by

$$f(x) = \begin{cases} \langle x, 1\rangle, & \text{if } x \neq m, \\ \langle 0, 0\rangle, & \text{if } x = m. \end{cases}$$

Then, f is an injective homomorphism, showing that $W' \in ISP(W)$. □

It may also be worthy to note that (21) is equivalent to the statement

$$-(a^{\smile} \odot -b) \text{ is the largest } x \text{ with } a \odot x \leqslant b,$$

i.e. $-(a^{\smile} \odot -b)$ is the (right) residuum of $\odot$. Hence, a more coherent way would be to introduce residua as e.g. in [22] as distinguished operators.

Let us formally define LCP algebras:

Definition 16. An ***LCP algebra*** is a system $(W, \wedge, \vee, 0, 1, ^{\smile}, \odot, 1')$ such that $(W, \wedge, \vee, 0, 1, ^{\smile})$ is an LC algebra, $(W, \wedge, \vee, 0, 1, \odot, 1')$ is an LP algebra and for all $a, b \in W$ the following holds:

(CP) $(a \odot b)^{\smile} = b^{\smile} \odot a^{\smile}$. $\square$

Note that

Lemma 27. *For every LCP algebra* $(W, \wedge, \vee, 0, 1, ^{\smile}, \odot, 1')$*,* $1'$ *is an equivalence element.*

Proof. We have to show that $1'$ is transitive and symmetric. Transitivity follows from $1' \odot 1' = 1'$, and symmetry can be shown as follows:

$$\begin{aligned}
1'^{\smile} &= 1'^{\smile} \odot 1', && \text{by } \textbf{(P.1)}\\
&= (1'^{\smile} \odot 1')^{\smile\smile}, && \text{by } \textbf{(C.1)}\\
&= (1'^{\smile} \odot 1')^{\smile}, && \text{by } \textbf{(CP)} \text{ and } \textbf{(C.1)}\\
&= 1'^{\smile\smile}, && \text{by } \textbf{(P.1)}\\
&= 1', && \text{by } \textbf{(C.1)}. \quad \square
\end{aligned}$$

Example 1. Let us consider a system $(W, \wedge, \vee, 0, 1, ^{\smile}, 1')$ such that $W = \{0, a, b, c, d, e, 1\}$, $1' = b$ and the operations $^{\smile}$ and $\odot$ are given in Tables 1 and 2 below, respectively.

x	$x^{\smile}$
0	0
a	a
b	b
c	d
d	c
e	e
1	1

Table 1. Converse $^{\smile}$ of W

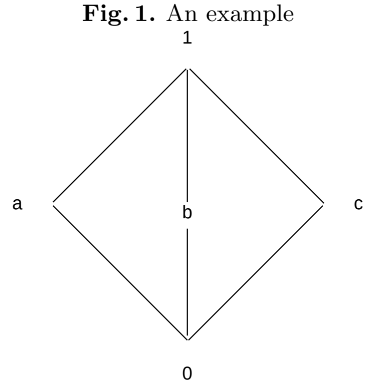
$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	c	d	e	1
b	0	a	b	c	d	e	1
c	0	c	c	c	e	e	c
d	0	d	d	0	d	0	d
e	0	e	e	0	e	0	e
1	0	1	1	c	d	e	1

Table 2. Product $\odot$ of W

By straightforward verification it is easy to check that W is an LCP algebra. Note that the underlying lattice is not distributive and product of W is non–commutative ($c \odot e = e \neq 0 = e \odot c$). $\square$

As before, we start our discussion on LCP algebras by defining *LCP frames*.

Fig. 1. An example



Definition 17. An ***LCP frame*** is a system $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ such that $(X, \leqslant_1, \leqslant_2, C)$ is an LC frame, $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ is an LP frame and moreover, for all $x, y, z \in X$,

(SCP) $Q(x, y, z) = S(C(y), C(x), C(z))$. $\square$

Definition 18. Let $(X, \preccurlyeq_1, \preccurlyeq_2, C, R, S, Q, I)$ be an LCP frame. A ***complex algebra of*** X is a structure $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, {}^{\curlyvee}, \otimes, \mathbf{1'})$ with operations defined by (3)–(6), (11), (15) and (16). $\square$

Theorem 9. *The complex algebra of an LCP frame is an LCP algebra.*

Proof. Let $(X, \preccurlyeq_1, \preccurlyeq_2, C, R, S, Q, I)$ be an LCP frame and let $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, {}^{\curlyvee}, \otimes, \mathbf{1'})$ be its complex algebra. From Theorems 4 and 6 it follows that $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, {}^{\curlyvee})$ is an LC algebra and $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ is an LP algebra. It suffices to show that **(CP)** holds in $L(X)$, i.e. $(A \otimes B)^{\curlyvee} = B^{\curlyvee} \otimes A^{\curlyvee}$ for all l–stable sets $A, B \subseteq X$.

($\subseteq$) Let $z \in (A \otimes B)^{\curlyvee}$. This means that $C(z) \in A \otimes B$, or equivalently, **(1)** $C(z) \in l(A \odot_Q B)$. Let $z' \in X$ be such that **(2)** $C(z) \leqslant_1 z'$. By **(MC.1)** and **(SC)** this is equivalent to **(3)** $z \leqslant_1 C(z')$. From **(1)** and **(2)** we get $z' \notin A \odot_Q B$. By the definition (13) this means that there exist $x, y \in X$ such that **(4)** $Q(x, y, z')$, **(5)** $x \in A$ and **(6)** $y \notin r(B)$. From **(6)**, there exists $y' \in X$ such that **(7)** $y \leqslant_2 y'$ and **(8)** $y' \in B$. Next, **(5)** implies **(9)** $C(x) \in A^{\curlyvee}$. Similarly, from **(8)** it follows $C(y') \in B^{\curlyvee}$, so **(10)** $C(y') \notin r(B^{\curlyvee})$. By **(MP.3)**, **(4)** and **(7)** imply $Q(x, y', z')$, which by **(SCP)** is equivalent to **(11)** $S(C(y'), C(x), C(z'))$. Hence, by **(9)** and **(10)** we get that $C(z') \notin B^{\curlyvee} \odot_S A^{\curlyvee}$. Since A and B are l–stable, by Lemma 11**(iii)** $A^{\curlyvee}$ and $B^{\curlyvee}$ are also l–stable. Hence, from Lemma 17, $B^{\curlyvee} \odot_S A^{\curlyvee} = B^{\curlyvee} \odot_Q A^{\curlyvee}$ which together with **(3)** gives $z \in l(B^{\curlyvee} \odot_Q A^{\curlyvee})$, that is $z \in B^{\curlyvee} \otimes A^{\curlyvee}$.

The proof of ($\supseteq$) is similar. ■

Let $(W, \wedge, \vee, 0, 1, {}^{\smile}, \odot, 1')$ be an LCP algebra. As before, by a *filter* (*ideal*) of W we mean a filter (ideal) of the underlying lattice $(W, \wedge, \vee, 0, 1)$. We will write $X(W)$ to denote the family of all maximal filter–ideal pairs of the lattice reduct of W.

Definition 19. Let $(W, \wedge, \vee, 0, 1, {}^{\smile}, \odot, 1')$ be an LCP algebra. A ***canonical frame of*** W is a structure $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, S^\star, Q^\star, I^\star)$ such that $\preccurlyeq_1$ and $\preccurlyeq_2$ are defined by (7) and (8), respectively, and relations $C^\star$, $R^\star$, $S^\star$, $Q^\star$ and $I^\star$ are defined by (12) and (17)–(20), respectively. □

Theorem 10. *The canonical frame of an LCP algebra is an LCP frame.*

Proof. Let $(X, \wedge, \vee, 0, 1, {}^{\smile}, \odot, 1')$ be an LCP algebra and let $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, S^\star, Q^\star, I^\star)$ be its canonical frame. By Theorem 4, $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star)$ is an LC frame. Next, by Theorem 7, $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ is an LP frame. It suffices to show that **(SCP)** holds.
Let $x, y, z \in X(W)$. We have the following equivalences:

$$\begin{array}{lll}
\multicolumn{3}{l}{S^\star(C^\star(y), C^\star(x), C^\star(z))} \\
\text{iff} & (\forall a, b \in W)\ a \odot b \in z_2^{\smile}\ \&\ b \in x_1^{\smile} \Rightarrow a \in y_2^{\smile} & \text{by (18)} \\
\text{iff} & (\forall a, b \in W)\ (a \odot b)^{\smile} \in z_2\ \&\ b^{\smile} \in x_1 \Rightarrow a^{\smile} \in y_2 & \text{by (10)} \\
\text{iff} & (\forall a, b \in W)\ b^{\smile} \odot a^{\smile} \in z_2\ \&\ b^{\smile} \in x_1 \Rightarrow a^{\smile} \in y_2 & \text{by axiom } \textbf{(CP)} \\
\text{iff} & Q^\star(x, y, z) & \text{by (19).}
\end{array}$$
■

We conclude the paper by stating the representability of LCP algebras.

Theorem 11. *Every LCP algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. Follows from Theorems 5, 8, 9 and 10. ■

7 Conclusions

In this paper we have studied not necessarily distributive lattices with the operators which are the abstract counterparts to the converse and composition of binary relations. On the algebraic side, we have presented relational representation theorems for these classes of algebras. The representation theorems are obtained by a suitable extensions of the Urquhart representation theorem for lattices [25]. However, here we stress the relational aspect of representability, and we omit the topological aspect. On the logical side, with every class of algebras studied in the paper we have associated an appropriate class of frames. These frames constitute a basis of a Kripke-style semantics for the logics whose algebraic semantics is determined by the classes of algebras presented in the paper. The representation theorems would enable us to prove completeness of the logics. For a detailed elaboration of the respective relational logics one can follow the developments in [1] and [23].

The further work is planned on the class of relation algebras based on double residuated lattices presented in [22]. The signature of such algebras extends the signature of LCP algebras with the residuation operators determined by the product, and with the sum and its dual residua. In these algebras complement operations are definable in terms of residua. Therefore the axioms of this class of algebras should include some counterparts to the De Morgan theorem K.

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Lattice–based relation algebras and their representability

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1 Introduction

The motivation for this paper comes from the following sources. First, one can observe that the two major concepts underlying the methods of reasoning with incomplete information are the concept of degree of truth of a piece of information and the concept of approximation of a set of information items. We shall refer to the theories employing the concept of degree of truth as to theories of fuzziness and to the theories employing the concept of approximation as to theories of roughness (see [5] for a survey). The algebraic structures relevant to these theories are residuated lattices ([6], [11], [15], [16], [21], [22]) and Boolean algebras with operators ([18], [20], [9], [10]), respectively. Residuated lattices provide an arithmetic of degrees of truth and Boolean algebras equipped with the appropriate operators provide a method of reasoning with approximately determined information. Both classes of algebras have a lattice structure as a basis. Second, both theories of fuzziness and theories of roughness develop generalizations of relation algebras to algebras of fuzzy relations [19] and algebras of rough relations ([3], [4], [8]), respectively. In both classes a lattice structure is a basis. Third, not necessarily distributive lattices with modal operators, which can be viewed as most elementary approximation operators, are recently developed in [23] (distributive lattices with operators are considered in [13] and [24]). With this background, our aim is this paper is to begin a systematic study of the classes of algebras that have the structure of a (not necessarily distributive) lattice and, moreover, in each class there are some operators added to the lattice which are relevant for binary relations. Our main interest is in developing relational representation theorems for the classes of lattices with operators under consideration. More precisely, we wish to guarantee that each algebra of our classes is isomorphic to an algebra of binary relations on a set. We prove the theorems of that form by suitably extending the Urquhart representation theorem for lattices ([25]) and the representation theorems presented in [1]. The classes defined in the paper are the parts which put together lead to what might be called lattice-based relation algebras. Our view is that these algebras would be the weakest structures relevant for binary relations. All the other algebras of binary relations considered in the literature would then be their extensions.

Throughout the paper we use the same symbol for denoting an algebra or a relational system and their universes.

2 Doubly ordered sets

Definition 1. Let X be a non–empty set and let $\leqslant_1$ and $\leqslant_2$ be two partial orderings in X. A structure $(X, \leqslant_1, \leqslant_2)$ is called a ***doubly ordered set*** iff for all $x, y \in X$, if $x \leqslant_1 y$ and $x \leqslant_2 y$ then $x = y$. $\square$

Definition 2. Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. We say that $A \subseteq X$ is $\leqslant_1$–***increasing*** (resp. $\leqslant_2$–***increasing***) whenever for all $x, y \in X$, if $x \in A$ and $x \leqslant_1 y$ (resp. $x \leqslant_2 y$), then $y \in A$. $\square$

For a doubly ordered set $(X, \leqslant_1, \leqslant_2)$, we define two mappings $l, r : 2^X \to 2^X$ by: for every $A \subseteq X$,

$$l(A) = \{x \in X : (\forall y \in X)\ x \leqslant_1 y \Rightarrow y \notin A\} \tag{1}$$

$$r(A) = \{x \in X : (\forall y \in X)\ x \leqslant_2 y \Rightarrow y \notin A\}. \tag{2}$$

Observe that mappings l and r can be expressed in terms of modal operators as follows: $l(A) = [\leqslant_1](-A)$ and $r(A) = [\leqslant_2](-A)$, where $-$ is the Boolean complement and $[\leqslant_i]$, $i = 1, 2$, are the necessity operators determined by relations $\leqslant_i$. Consequently, r and l are intuitionistic–like negations.

Definition 3. Given a doubly ordered set $(X, \leqslant_1, \leqslant_2)$, a subset $A \subseteq X$ is called ***l–stable*** (resp. ***r–stable***) iff $l(r(A)) = A$ (resp. $r(l(A)) = A$). $\square$

The family of all l-stable (resp. r–stable) subsets of X will be denoted by $L(X)$ (resp. $R(X)$).

Recall the following notion from e.g. [7]:

Definition 4. Let $(X, \leqslant_1)$ and $(Y, \leqslant_2)$ be partially ordered sets and let $f : X \to Y$ and $g : Y \to X$. We say that f and g are ***Galois connection*** iff for all $x, y \in X$

$$x \leqslant_1 g(y) \text{ iff } y \leqslant_2 f(x). \quad \square$$

Lemma 1. [23] *For any doubly ordered set $(X, \leqslant_1, \leqslant_2)$ and for any $A \subseteq X$,*

(i) *$l(A)$ is $\leqslant_1$–increasing*
(ii) *$r(A)$ is $\leqslant_2$–increasing*
(iii) *if A is $\leqslant_1$–increasing, then $r(A) \in R(X)$*
(iv) *if A is $\leqslant_2$–increasing, then $l(A) \in L(X)$*
(v) *if $A \in L(X)$, then $r(A) \in R(X)$*
(vi) *if $A \in R(X)$, then $l(A) \in L(X)$*
(vii) *if $A, B \in L(X)$, then $r(A) \cap r(B) \in R(X)$.* ■

Lemma 2. [23] *The family of $\leqslant_i$–increasing sets, $i = 1, 2$, forms a distributive lattice, where join and meet are union and intersection of sets.* ■

Lemma 3. [23] *For every doubly ordered set $(X, \leqslant_1, \leqslant_2)$, the mappings l and r form a Galois connection between the lattice of $\leqslant_1$–increasing subsets of X and the lattice of $\leqslant_2$–increasing subsets of X.* ■

In other words, Lemma 3 says that for any $A \in L(X)$ and for any $B \in R(X)$, $A \subseteq l(B)$ iff $B \subseteq r(A)$.

Lemma 4. *For every doubly ordered set $(X, \leqslant_1, \leqslant_2)$ and for every $A \subseteq X$,*

(i) *$l(r(A)) \in L(X)$ and $r(l(A)) \in R(X)$*
(ii) *if A is $\leqslant_1$–increasing, then $A \subseteq l(r(A))$*
(iii) *if A is $\leqslant_2$–increasing, then $A \subseteq r(l(A))$.*

Proof. Direct consequence of Lemmas 1 and 3. ■

Lemma 4 immediately implies:

Corollary 1. *For every doubly ordered set $(X, \leqslant_1, \leqslant_2)$ and for every $A \subseteq X$,*

(i) *if $A \in L(X)$, then $A \subseteq l(r(A))$*
(ii) *if $A \in R(X)$, then $A \subseteq r(l(A))$.* ■

Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. Define two binary operations in 2^X: for all $A, B \subseteq X$,

$$A \sqcap B = A \cap B \tag{3}$$

$$A \sqcup B = l(r(A) \cap r(B)). \tag{4}$$

Observe that $\sqcup$ is defined from $\sqcap$ resembling a De Morgan law with two different negations.
Moreover, put

$$\mathbf{0} = \emptyset. \tag{5}$$
$$\mathbf{1} = X \tag{6}$$

Lemma 5. [25] *For any doubly ordered set* $(X, \leqslant_1, \leqslant_2)$*, the system* $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ *is a lattice.* ■

Definition 5. Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. The lattice $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ is called the ***complex algebra of*** X. □

3 Urquhart representation of lattices

Let $(W, \wedge, \vee, 0, 1)$ be a bounded lattice.

Definition 6. A ***filter-ideal pair*** of a lattice W is a pair $x = (x_1, x_2)$ such that x_1 is a filter of W, x_2 is an ideal of W and $x_1 \cap x_2 = \emptyset$. □

The family of all filter–ideal pairs of a lattice W will be denoted by $FIP(W)$.

Let us define the following two quasi ordering relations on $FIP(W)$: for any $(x_1, x_2), (y_1, y_2) \in FIP(W)$,

$$(x_1, x_2) \preccurlyeq_1 (y_1, y_2) \quad \text{iff} \quad x_1 \subseteq y_1 \tag{7}$$
$$(x_1, x_2) \preccurlyeq_2 (y_1, y_2) \quad \text{iff} \quad x_2 \subseteq y_2. \tag{8}$$

Next, define

$$(x_1, x_2) \preccurlyeq (y_1, y_2) \quad \text{iff} \quad (x_1, x_2) \preccurlyeq_1 (y_1, y_2) \;\&\; (x_1, x_2) \preccurlyeq_2 (y_1, y_2).$$

We say that $(x_1, x_2) \in FIP(W)$ is *maximal* iff it is maximal wrt $\preccurlyeq$. We will write $X(W)$ to denote the family of all maximal filter–ideal pairs of the lattice W.

Observe that $X(W)$ is a binary relation on 2^W.

Proposition 1. [25] *Let* W *be a bounded lattice. For any* $(x_1, x_2) \in FIP(W)$ *there exists* $(y_1, y_2) \in X(W)$ *such that* $(x_1, y_1) \preccurlyeq (y_1, y_2)$. ■

For any $(x_1, x_2) \in FIP(W)$, the maximal filter–ideal pair (y_1, y_2) such that $(x_1, x_2) \preccurlyeq (y_1, y_2)$ will be referred to as an *extension* of (x_1, x_2).

Definition 7. Let $(W, \wedge, \vee, 0, 1)$ be a bounded lattice. The ***canonical frame of*** W is the structure $(X(W), \preccurlyeq_1, \preccurlyeq_2)$. $\Box$

Lemma 6. *For every bounded lattice* W*,*

(i) *its canonical frame* $(X(W), \preccurlyeq_1, \preccurlyeq_2)$ *is a doubly ordered set*
(ii) *for all* $x, y \in X(W)$*, if* $x \preccurlyeq_1 y$ *and* $x \preccurlyeq_2 y$*, then* $x = y$*.* $\blacksquare$

Consider the complex algebra $(L(X(W)), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ of the canonical frame of a lattice $(W, \wedge, \vee, 0, 1)$. Observe that $L(X(W))$ is an algebra of subrelations of $X(W)$.

Let us define the mapping $h : W \to 2^{X(W)}$ as follows: for every $a \in W$,

$$h(a) = \{x \in X(W) : a \in x_1\}. \tag{9}$$

Theorem 1. [25] *For every lattice* $(W, \wedge, \vee, 0, 1)$ *the following assertions hold:*

(i) *For every* $a \in W$*,* $r(h(a)) = \{x \in X(W) : a \in x_2\}$*.*
(ii) $h(a)$ *is* l*–stable for every* $a \in W$*.*
(iii) h *is a lattice embedding.*

Proof. By way of example we prove **(iii)**.
We show that h is injective. Assume that for some $a, b \in W$, $h(a) = h(b)$. It follows that for every $x \in X(W)$, $a \in x_1$ iff $b \in x_1$. In particular, if $x_1 = [a) = \{z \in W : a \leqslant z\}$, then clearly $a \in [a)$, and also by the assumption $b \in [a)$. Hence $a \leqslant b$. Similarly, if $x_1 = [b)$, then $b \leqslant a$. We conclude that $a = b$.
Now we show that h preserves the operations. By way of example we prove that $h(a) \sqcup h(b) = h(a \vee b)$. Indeed. for every $a, b \in W$,

$$\begin{aligned}
&h(a) \sqcup h(b) \\
&\quad = l(r(\{x \in X(W) : a \in x_1\}) \cap r(\{x \in X(W) : b \in x_1\})) \\
&\quad = l(\{x \in X(W) : a \in x_2\} \cap \{x \in X(W) : b \in x_2\}) && \text{from } \mathbf{(i)} \\
&\quad = l(\{x \in X(W) : a \vee b \in x_2\}) && \text{since } x_2 \text{ is an ideal} \\
&\quad = lr(\{x \in X(W) : a \vee b \in x_1\}) && \text{from } \mathbf{(i)} \\
&\quad = lr(h(a \vee b)) && \text{the definition of } h \\
&\quad = h(a \vee b) && \text{from } \mathbf{(ii)}. \qquad \Box
\end{aligned}$$

Theorem 2 (Representation theorem for lattices). *Every bounded lattice is isomorphic to a subalgebra of the complex algebra of its canonical frame.* $\Box$

4 LC algebras

An LC algebra is a bounded lattice with an additional unary operator which is an abstract counterpart of the relational converse.

Definition 8. An ***LC algebra*** is a structure $(W, \wedge, \vee, 0, 1, ^{\smile})$ such that $(W, \wedge, \vee, 0, 1)$ is a bounded lattice and $^{\smile}$ is a unary operator on W such that for all $a, b \in W$,

(C.1) $a^{\smile\smile} = a$
(C.2) $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$. □

For an LC algebra W and any $a \in W$, $a^{\smile}$ is called a *converse of a*.

The following lemma gives the basic properties of the converse operation.

Lemma 7. *Let $(W, \wedge, \vee, 0, 1, ^{\smile})$ be an LC algebra. Then the following assertions hold:*

(i) $0^{\smile} = 0$, $1^{\smile} = 1$.
(ii) *for all $a, b \in W$, $a \leqslant b$ implies $a^{\smile} \leqslant b^{\smile}$*
(iii) *for all $a, b \in W$, $(a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}$.*

Proof. The proof of **(i)** is similar to the one presented in [2]. Namely, by Definition 8 we have: $0^{\smile} = 0 \vee 0^{\smile} = 0^{\smile\smile} \vee 0^{\smile} = (0^{\smile} \vee 0)^{\smile} = 0^{\smile\smile} = 0$. Analogously, $1 = 1 \vee 1^{\smile} = 1^{\smile\smile} \vee 1^{\smile} = (1^{\smile} \vee 1)^{\smile} = 1^{\smile}$.

(ii) Assume that $a \leqslant b$. Then $a \vee b = b$, so $(a \vee b)^{\smile} = b^{\smile}$. By axiom **(C.2)**, $a^{\smile} \vee b^{\smile} = b^{\smile}$, so $a^{\smile} \leqslant b^{\smile}$

(iii) **(⩽)** Let $a, b \in W$. Since $a \wedge b \leqslant a$, by **(ii)** we get $(a \wedge b)^{\smile} \leqslant a^{\smile}$. Similarly, $(a \wedge b)^{\smile} \leqslant b^{\smile}$, which yields $(a \wedge b)^{\smile} \leqslant a^{\smile} \wedge b^{\smile}$.

(⩾) Since $a^{\smile} \wedge b^{\smile} \leqslant a^{\smile}$, we have $(a^{\smile} \wedge b^{\smile})^{\smile} \leqslant a^{\smile\smile} = a$ by **(ii)** and **(C.1)**. Analogously, $(a^{\smile} \wedge b^{\smile})^{\smile} \leqslant b$, so $(a^{\smile} \wedge b^{\smile})^{\smile} \leqslant a \wedge b$. Applying again **(C.1)** and **(ii)** we get $a^{\smile} \wedge b^{\smile} = (a^{\smile} \wedge b^{\smile})^{\smile\smile} \leqslant (a \wedge b)^{\smile}$. ■

Given an LC algebra $(W, \wedge, \vee, 0, 1, ^{\smile})$, by a *filter* (resp. *ideal*) of W we mean a filter (resp. ideal) of the underlying lattice $(W, \wedge, \vee, 0, 1)$.

For any $A \subseteq W$, by $A^{\smile}$ we will denote the set:

$$A^{\smile} = \{a^{\smile} \in W : a \in A\}. \tag{10}$$

We have the following:

Lemma 8. *Let* $(W, \wedge, \vee, 0, 1, ^{\smile})$ *be an LC algebra. Then the following assertions hold for all* $A, B \subseteq W$*:*

(i) $A^{\smile} = \{a \in W : a^{\smile} \in A\}$
(ii) $A^{\smile\smile} = A$
(iii) $A \subseteq B$ *iff* $A^{\smile} \subseteq B^{\smile}$
(iv) $(-A)^{\smile} = -(A^{\smile})$
(v) $(A \cup B)^{\smile} = A^{\smile} \cup B^{\smile}$
(vi) $(A \cap B)^{\smile} = A^{\smile} \cap B^{\smile}$.

Proof. By way of example we show **(ii)** and **(iii)**.
(ii) Let $a \in W$. Then $a \in A^{\smile\smile}$ iff $a^{\smile} \in A^{\smile}$ iff $a^{\smile\smile} \in A$ iff $a \in A$.
(iii) **(⇒)** Let $A, B \subseteq W$ be such that $A \subseteq B$ and let $a \in A^{\smile}$. Hence, by definition (10), $a^{\smile} \in A$, which by assumption implies $a^{\smile} \in B$.
(⇐) Assume that $A^{\smile} \subseteq B^{\smile}$ and $a \in A$. By **(C.1)**, $a^{\smile\smile} \in A$, so $a^{\smile} \in A^{\smile}$, which by assumption gives $a^{\smile} \in B^{\smile}$. It follows that $a^{\smile\smile} \in B$, and hence, $a \in B$. ∎

Lemma 9. *Let* $(W, \wedge, \vee, 0, 1, ^{\smile})$ *be an LC algebra and let* $A \subseteq W$*. Then the following assertions hold:*

(i) *If* A *is a filter of* W*, then so is* $A^{\smile}$
(ii) *If* A*. is an ideal of* W*, then so is* $A^{\smile}$.

Proof.
(i) Let A be a filter of W and $a, b \in W$ such that $a \leqslant b$ and $a \in A^{\smile}$. Then $a^{\smile} \in A$, and, by Lemma 7**(ii)** we also have $a^{\smile} \leqslant b^{\smile}$. This implies, $b^{\smile} \in A$, and thus, $b \in A^{\smile}$.
Let $a, b \in A^{\smile}$. This means that $a^{\smile} \in A$ and $b^{\smile} \in A$, and therefore, $a^{\smile} \wedge b^{\smile} \in A$ since A is a filter. By Lemma 7**(iii)**, $a^{\smile} \wedge b^{\smile} = (a \wedge b)^{\smile}$, so that $(a \wedge b)^{\smile} \in A = A^{\smile\smile}$ by Lemma 8**(ii)**. Then $(a \wedge b)^{\smile\smile} \in A^{\smile}$, or equivalently, $a \wedge b \in A^{\smile}$.
(ii) Let A be an ideal of W and let $a, b \in W$. Assume that $b \in A^{\smile}$ and $a \leqslant b$. Then $b^{\smile} \in A$ and by Lemma 7**(ii)**, $a^{\smile} \leqslant b^{\smile}$. Hence $a^{\smile} \in A$, so $a \in A^{\smile}$.
Let $a, b \in A^{\smile}$. Then $a^{\smile} \in A$ and $b^{\smile} \in A$. Since A is an ideal, $a^{\smile} \vee b^{\smile} \in A$. By axiom **(C.1)**, $a^{\smile} \vee b^{\smile} = (a \vee b)^{\smile}$. Hence $(a \vee b)^{\smile} \in A$, so $a \vee b \in A^{\smile}$. ∎

4.1 LC frames

Definition 9. An ***LC frame*** is a relational system $(X, \leqslant_1, \leqslant_2, C)$ such that $(X, \leqslant_1, \leqslant_2)$ is a doubly ordered set and C is a mapping $C : X \to X$ satisfying the following conditions for all $x, y \in X$:

(MC.1) $x \leqslant_1 y$ implies $C(x) \leqslant_1 C(y)$
(MC.2) $x \leqslant_2 y$ implies $C(x) \leqslant_2 C(y)$
(SC) $C(C(x)) = x$. $\square$

Given an LC frame $(X, \leqslant_1, \leqslant_2, C)$ let us define a mapping ${}^{\curlyvee} : 2^X \to 2^X$ as follows: for every $A \subseteq X$,

$$A^{\curlyvee} = \{C(x) : x \in A\}. \tag{11}$$

The following two lemmas present some properties of ${}^{\curlyvee}$.

Lemma 10. *Let $(X, \leqslant_1, \leqslant_2, C)$ be an LC frame and let ${}^{\curlyvee}$ be defined by (11). Then for all $A, B \subseteq X$,*

(i) $A^{\curlyvee} = \{x \in X : C(x) \in A\}$
(ii) $A^{\curlyvee\curlyvee} = A$
(iii) $A \subseteq B$ *implies* $A^{\curlyvee} \subseteq B^{\curlyvee}$
(iv) $(A \cap B)^{\curlyvee} = A^{\curlyvee} \cap B^{\curlyvee}$.

Proof. By way of example we show **(ii)** and **(iv)**.

(ii) Let $x \in W$. By **(SC)** and the definition (11) we have the following equivalences: $x \in A$ iff $C(C(x)) \in A$ iff $C(x) \in A^{\curlyvee}$ iff $x \in A^{\curlyvee\curlyvee}$.

(iv)($\subseteq$) Since $A \cap B \subseteq A$, by **(iii)** it follows that $(A \cap B)^{\curlyvee} \subseteq A^{\curlyvee}$. Similarly, $(A \cap B)^{\curlyvee} \subseteq B^{\curlyvee}$. Then $(A \cap B)^{\curlyvee} \subseteq A^{\curlyvee} \cap B^{\curlyvee}$.

($\supseteq$) Since $A^{\curlyvee} \cap B^{\curlyvee} \subseteq A^{\curlyvee}$, from **(iii)** and **(ii)** it follows $(A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee} \subseteq A^{\curlyvee\curlyvee} = A$. Also, $(A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee} \subseteq B$. Then $(A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee} \subseteq A \cap B$, so again by **(ii)** and **(iii)**, $A^{\curlyvee} \cap B^{\curlyvee} = (A^{\curlyvee} \cap B^{\curlyvee})^{\curlyvee\curlyvee} \subseteq (A \cap B)^{\curlyvee}$. $\blacksquare$

Lemma 11. *Let $(X, \leqslant_1, \leqslant_2, C)$ be an LC frame and let ${}^{\curlyvee}$ be defined by (11). Then for all $A, B \subseteq X$,*

(i) $l(A^{\curlyvee}) = l(A)^{\curlyvee}$
(ii) $r(A^{\curlyvee}) = r(A)^{\curlyvee}$.
(iii) *if A is l–stable, then so is $A^{\curlyvee}$.*

Proof. By way of example we show **(i)** and **(iii)**.

(i) **($\subseteq$)** Let $x \notin l(A)^{\curlyvee}$. By the definition (11), this means that $C(x) \notin l(A)$, so there exists $y \in X$ such that **(i.1)** $C(x) \leqslant_1 y$, and **(i.2)** $y \in A$. By **(MC.1)**, **(i.1)** implies $C(C(x)) \leqslant_1 C(y)$, so by **(SC)** we get **(i.3)** $x \leqslant_1 C(y)$. Next, **(i.2)** and **(SC)** imply $C(C(y)) \in A$, whence $C(y) \in A^{\curlyvee}$, which together with **(i.3)** implies $x \notin l(A^{\curlyvee})$.

($\supseteq$) can be proved in the similar way.

(iii) Let A be l–stable. By **(i)** and **(ii)**, $l(r(A^{\curlyvee})) = l(r(A)^{\curlyvee}) = l(r(A))^{\curlyvee} = A^{\curlyvee}$, so $A^{\curlyvee}$ is l–stable. $\blacksquare$

4.2 Complex algebras of LC frames

Definition 10. Let $(X, \leqslant_1, \leqslant_2, C)$ be an LC frame. By the ***complex algebra of*** X we mean a structure $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, ^{\curlyvee})$ with the operations defined by (3), (4), (11) and the constants defined by (5) and (6). □

Theorem 3. *The complex algebra of an LC frame is an LC algebra.*

Proof. From Lemma 10**(ii)**, $A^{\curlyvee\curlyvee} = A$, so it suffices to show that $(A \sqcup B)^{\curlyvee} = A^{\curlyvee} \sqcup B^{\curlyvee}$. For every $x \in X$,

$x \in (A \sqcup B)^{\curlyvee}$	iff	$C(x) \in A \sqcup B$	by the definition of $^{\curlyvee}$
	iff	$C(x) \in l(r(A) \cap r(B))$	by the definition of $\sqcup$
	iff	$x \in l(r(A) \cap r(B))^{\curlyvee}$	by the definition of $^{\curlyvee}$
	iff	$x \in l((r(A) \cap r(B))^{\curlyvee})$	by Lemma 11**(i)**
	iff	$x \in l(r(A)^{\curlyvee} \cap r(B)^{\curlyvee})$	by Lemma 10**(iv)**
	iff	$x \in l(r(A^{\curlyvee}) \cap r(B^{\curlyvee}))$	by Lemma 11**(ii)**
	iff	$x \in A^{\curlyvee} \sqcup B^{\curlyvee}$. ■	

4.3 Canonical frames of LC algebras

Let $(W, \wedge, \vee, 0, 1, ^{\smile})$ be an LC algebra. As usual, $FIP(W)$ and $X(W)$ denote the family of all filter–ideal pairs (resp. maximal filter–ideal pairs) of W.

First, observe the following

Lemma 12. $(x_1, x_2) \in FIP(W)$ *iff* $(x_1^{\smile}, x_2^{\smile}) \in FIP(W)$.

Proof. **(⇒)** Let $(x_1, x_2) \in FIP(W)$. From Lemma 9 it follows that $x_1^{\smile}$ is a filter of W and $x_2^{\smile}$ is an ideal of W. Note that $\emptyset^{\smile} = \emptyset$. Then, by Lemma 8**(vi)** we get that $x_1^{\smile} \cap x_2^{\smile} = \emptyset$, so $(x_1^{\smile}, x_2^{\smile}) \in FIP(W)$

(⇐) Let $x_1, x_2 \subseteq W$ be such that $(x_1^{\smile}, x_2^{\smile}) \in FIP(W)$. Then, by Lemmas 8**(ii)** and 9, $x_1 = x_1^{\smile\smile}$ is a filter of W and $x_2 = x_2^{\smile\smile}$ is an ideal of W. Next, from Lemma 8**(vi)**, $(x_1 \cap x_2)^{\smile} = \emptyset$, so $x_1 \cap x_2 = (x_1 \cap x_2)^{\smile\smile} = \emptyset$. Whence $(x_1, x_2) \in FIP(W)$. ■

Let us now define a mapping $C^{\star} : FIP(W) \to FIP(W)$ as follows: for every $x \in FIP(W)$,

$$C^{\star}(x) = (x_1^{\smile}, x_2^{\smile}). \tag{12}$$

Lemma 13. *If x is a maximal filter–ideal pair of W, then so is $C^{\star}(x)$.*

Proof. Let $x=(x_1,x_2)\in FIP(W)$. Assume that $(x_1^{\smile},x_2^{\smile})$ is not maximal. By Proposition 1, it can be extended to the maximal filter–ideal pair, say $y=(y_1,y_2)$. Then $x_1^{\smile}\subseteq y_1$, $x_2^{\smile}\subseteq y_2$ and $(x_1^{\smile},x_2^{\smile})\neq(y_1,y_2)$. By Lemma 8**(ii)** and 8**(iii)** we get $x_1\subseteq y_1^{\smile}$, $x_2\subseteq y_2^{\smile}$ and $(x_1,x_2)\neq(y_1^{\smile},y_2^{\smile})$, which means that (x_1,x_2) is not a maximal filter–ideal pair. ■

Definition 11. Let $(W,\wedge,\vee,0,1,^{\smile})$ be an LC algebra. A ***canonical frame*** of W is a structure $(X(W),\preccurlyeq_1,\preccurlyeq_2,C^\star)$, where $\preccurlyeq_1$, $\preccurlyeq_2$ and $C^\star$ are defined by (7), (8) and (12), respectively. □

Theorem 4. *The canonical frame of a LC algebra is an LC frame.*

Proof. Let $x,y\in X(W)$ and assume that $x\preccurlyeq_1 y$. This means that $x_1\subseteq y_1$. By Lemma 8**(iii)**, $x_1^{\smile}\subseteq y_1^{\smile}$, so $C^\star(x)\preccurlyeq_1 C^\star(y)$. Hence **(MC.1)** holds. In the analogous way we can show that **(MC.2)** holds. Finally, let $x=(x_1,x_2)\in X(W)$. Then we have $C^\star(C^\star(x))=(x_1^{\smile\smile},x_2^{\smile\smile})=(x_1,x_2)=x$ by Lemma 8**(ii)**, so the condition **(SC)** also holds. ■

4.4 Relational representation of LC algebras

In this section we conclude our discussion of LC algebras by showing their relational representability.

Theorem 5 (Representation theorem for LC algebras). *Every LC algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. Let $(W,\wedge,\vee,0,1,^{\smile})$ be an LC algebra, $(X(W),\preccurlyeq_1,\preccurlyeq_2,C^\star)$ be the canonical frame of W and let $(L(X(W)),\sqcap,\sqcup,\mathbf{0},\mathbf{1},^{\curlyvee})$ be the complex algebra of the canonical frame of W. By Theorems 3 and 4 it follows that $L(X(W))$ is an LC algebra, so it suffices to show that W is isomorphic to a subalgebra of $L(X(W))$.
Let the mapping $h:W\rightarrow 2^{X(W)}$ be defined as in (9), i.e. $h(a)=\{x\in X(W): a\in x_1\}$, $a\in W$. We show that for every $a\in W$, $h(a^{\smile})=h(a)^{\curlyvee}$. For every $x\in X(W)$ and for every $a\in W$ we have: $x\in h(a^{\smile})$ iff $a^{\smile}\in x_1$ iff $a\in x_1^{\smile}$ iff $C^\star(x)\in h(a)$ iff $x\in h(a)^{\curlyvee}$. ■

5 LP algebras

LP algebras are a join of a not necessary distributive bounded lattice with a monoid. The monoid product operation is an abstract counterpart to the relational composition and the unit element of the monoid corresponds to the identity relation.

Definition 12. An ***LP algebra*** is a structure $(W, \wedge, \vee, 0, 1, \odot, 1')$ such that $(W, \wedge, \vee, 0, 1)$ is a bounded lattice and for all $a, b, c \in W$,

(P.1) $a \odot 1' = 1' \odot a = a$
(P.2) $a \odot (b \odot c) = (a \odot b) \odot c$
(P.3) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$
(P.4) $(a \vee b) \odot c = (a \odot c) \vee (b \odot c)$. □

Lemma 14. *Let $(W, \wedge, \vee, 0, 1, \odot, 1')$ be an LP algebra. For all $a, b, c \in W$, if $a \leqslant b$, then*

(i) $c \odot a \leqslant c \odot b$
(ii) $a \odot c \leqslant b \odot c$.

Proof.

(i) Let $a \leqslant b$. Then $a \vee b = b$, so $c \odot (a \vee b) = c \odot b$. Hence, by axiom **(P.3)**, we get $(c \odot a) \vee (c \odot b) = c \odot b$, which implies $c \odot a \leqslant c \odot b$.
(ii) This can be proved in an analogous way. ■

5.1 LP frames

In this section we follow the developments of Allwein and Dunn ([1]). However, the difference is that here we consider an abstract notion of an LP frame, not only the canonical frame of an LP algebra.

Definition 13. An ***LP frame*** is a relational system $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ such that $(X, \leqslant_1, \leqslant_2)$ is a doubly ordered set, R, S, Q are ternary relations on X and $I \subseteq X$ is an unary relation on X such that the following conditions are satisfied: for all $x, x', y, y', z, z' \in X$,

A. Monotonicity conditions
(MP.1) $R(x,y,z) \;\&\; x' \leqslant_1 x \;\&\; y' \leqslant_1 y \;\&\; z \leqslant_1 z' \Rightarrow R(x', y', z')$
(MP.2) $S(x,y,z) \;\&\; x \leqslant_2 x' \;\&\; y' \leqslant_1 y \;\&\; z' \leqslant_2 z \Rightarrow S(x', y', z')$
(MP.3) $Q(x,y,z) \;\&\; x' \leqslant_1 x \;\&\; y \leqslant_2 y' \;\&\; z' \leqslant_2 z \Rightarrow Q(x', y', z')$
(MP.4) $I(x) \;\&\; x \leqslant_1 x' \Rightarrow I(x')$

B. Stability conditions
(SP.1) $R(x,y,z) \Rightarrow \exists x'' \in X \; (x \leqslant_1 x'' \;\&\; S(x'', y, z))$
(SP.2) $R(x,y,z) \Rightarrow \exists y'' \in X \; (y \leqslant_1 y'' \;\&\; Q(x, y'', z))$
(SP.3) $S(x,y,z) \Rightarrow \exists z'' \in X \; (z \leqslant_2 z'' \;\&\; R(x, y, z''))$
(SP.4) $Q(x,y,z) \Rightarrow \exists z'' \in X \; (z \leqslant_2 z'' \;\&\; R(x, y, z''))$
(SP.5) $\exists u \in X (R(x,y,u) \;\&\; Q(x', u, z)) \Rightarrow \exists w \in X (R(x', x, w) \;\&\; S(w, y, z))$
(SP.6) $\exists u \in X (R(x,y,u) \;\&\; S(u, z, z')) \Rightarrow \exists w \in X (R(y, z, w) \;\&\; Q(x, w, z'))$
(SP.7) $I(x) \;\&\; (R(x,y,z)$ or $R(y,x,z)) \Rightarrow y \leqslant_1 z$
(SP.8) $\exists u \in X (I(u) \;\&\; S(u, x, x))$
(SP.9) $\exists u \in X (I(u) \;\&\; Q(x, u, x))$. □

Lemma 15. *For every LP frame* $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ *and every l–stable subset* $A \subseteq X$ *it holds for all* $x, y, z \in X$*:*

$$I(x) \ \& \ (y \in A) \ \& \ (R(x, y, z) \text{ or } R(y, x, z)) \ \Rightarrow \ z \in A.$$

Proof.

Let $x, y, z \in X$ and assume that **(1)** $I(x)$ **(2)** $y \in A$ and **(3)** $R(x, y, z)$ or $R(y, x, z)$. By **(SP.7)**, we get from **(1)** and **(3)** that **(4)** $y \leqslant_1 z$. Since A is l–stable, by Lemma 1**(i)** it is $\leqslant_1$–increasing, so **(2)** and **(4)** imply $z \in A$. ■

For an LP frame $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$, let us define two mappings $\odot_Q$, $\odot_S$: $2^X \times 2^X \to 2^X$ by: for all $A, B \subseteq X$,

$$A \odot_Q B = \{z \in X : \forall x, y \in X \ (Q(x, y, z) \ \& \ x \in A \ \Rightarrow \ y \in r(B))\} \qquad (13)$$

$$A \odot_S B = \{z \in X : \forall x, y \in X \ (S(x, y, z) \ \& \ y \in B \ \Rightarrow \ x \in r(A)\}. \qquad (14)$$

Lemma 16. *For every LP frame* $(X, \leqslant_1, \leqslant_2, R, S, C, I)$ *and for all l–stable subsets* $A, B \subseteq X$*,* $A \odot_S B = A \odot_Q B$*.*

Proof. **(⊆)** Let $z \in X$ and assume that $z \notin A \odot_Q B$. We show that $z \notin A \odot_S B$. By assumption, there exist $x, y \in X$ such that **(1)** $Q(x, y, z)$, **(2)** $x \in A$ and **(3)** $y \notin r(B)$. From **(3)**, there is $y' \in X$ such that **(4)** $y \leqslant_2 y'$ and **(5)** $y' \in B$. Moreover, **(2)** implies **(6)** $x \notin r(A)$. Next, from **(1)** and **(4)** we get by **(MP.3)** that $Q(x, y', z)$, which by **(SP.4)** implies that there exists $z' \in X$ such that **(7)** $z \leqslant_2 z'$ and **(8)** $R(x, y', z')$. Furthermore, applying **(SP.1)**, we get from **(8)** that there exists $x' \in X$ such that **(9)** $x \leqslant_1 x'$ and **(10)** $S(x', y', z')$. Also, by **(MP.2)**, from **(7)** and **(10)** it follows that **(11)** $S(x', y', z)$. Therefore, we have that for some $x', y' \in X$, **(11)**, **(5)** and **(6)** hold, which means that $x \notin A \odot_S B$.

The proof of **(⊇)** is similar. ■

Lemma 17. *For any* $A, B \subseteq X$*,*

- **(i)** $A \odot_Q B$ *and* $A \odot_S B$ *are* $\leqslant_2$*–increasing*
- **(ii)** *if* A *and* B *are l–stable, then* $A \odot_Q B$ *and* $A \odot_S B$ *are r–stable.*

Proof.

(i) Let $A, B \subseteq X$ and suppose that $A \odot_Q B$ is not $\leqslant_2$–increasing. Then, by Definition 2, there exist $x, y \in X$ such that **(i.1)** $x \in A \odot_Q B$, **(i.2)** $x \leqslant_1 y$ and **(i.3)** $y \notin A \odot_Q B$. By the definition (13), **(i.3)** means that there exist $u, w \in X$ such that **(i.4)** $Q(u, w, y)$, **(i.5)** $u \in A$ and **(i.6)** $w \notin r(B)$. However, by **(MP.3)**, **(i.2)** and **(i.4)** imply $Q(u, w, x)$, which together with **(i.5)** and **(i.6)** gives $x \notin A \odot_Q B$ – a contradiction with **(i.1)**.
In the similar way one can show that $A \odot_S B$ is $\leqslant_2$–increasing.

(ii) Let $A, B \subseteq X$ be l–stable sets. By **(i)**, $A \odot_Q B$ is $\leqslant_2$–increasing. Therefore, by Lemma 4**(iii)**, $A \odot_Q B \subseteq r(l(A \odot_Q B))$, so it suffices to show that $r(l(A \odot_Q B)) \subseteq A \odot_Q B$.
Let $z \in X$ and assume that $z \notin A \odot_Q B$. We show that **(ii.1)** $z \notin r(l(A \odot_Q B))$. By the definition (13) of $\odot_Q$, there exist $x, y \in X$ such that **(ii.2)** $Q(x, y, z)$, **(ii.3)** $x \in A$ and **(ii.4)** $y \notin r(B)$. From **(ii.4)**, there exists $y' \in X$ such that **(ii.5)** $y \leqslant_2 y'$ and **(ii.6)** $y' \in B$. By **(MP.3)**, **(ii.2)** and **(ii.5)** imply $Q(x, y', z)$, which by **(SP.4)** gives that there exists $z' \in X$ such that **(ii.7)** $z \leqslant_2 z'$ and **(ii.8)** $R(x, y', z')$.
Now we show that **(ii.9)** $z' \in l(A \odot_Q B)$. Let $z'' \in X$ and assume that **(ii.10)** $z' \leqslant_1 z''$. Hence, by **(MP.1)**, from **(ii.8)** we get that $R(x, y', z'')$, which by **(SP.2)** implies that there exists $y'' \in X$ such that **(ii.11)** $y' \leqslant_1 y''$ and **(ii.12)** $Q(x, y'', z'')$. Next, **(ii.10)** and **(ii.12)** imply **(ii.13)** $Q(x, y'', z')$ by **(MP.3)**. Furthermore, since B is l–stable, it is $\leqslant_1$–increasing, so from **(ii.6)** and **(ii.11)**, $y'' \in B$. It follows that **(ii.14)** $y'' \notin r(B)$. Then we have that for some $x, y'' \in X$, **(ii.3)**, **(ii.13)** and **(ii.14)** hold, which means that $z' \notin A \odot_Q B$. Since z'' was an arbitrary element satisfying **(ii.10)**, we obtain that for any $z'' \in X$, $z' \leqslant_1 z''$ implies $z' \notin A \odot_Q B$, so **(ii.9)** follows. Finally, from **(ii.7)** and **(ii.9)** we obtain $z \notin rl(A \odot_Q B)$.
Proceeding in the similar way we can show that $A \odot_S B$ is r–stable. ∎

Let us define a mapping $\otimes : 2^X \times 2^X \to 2^X$ as follows: for all $A, B \subseteq X$,

$$A \otimes B = l(A \odot_Q B). \tag{15}$$

Lemma 18. *Let $A, B \subseteq X$. If A and B are l–stable sets, then so is $A \otimes B$.*

Proof. By Lemma 17**(ii)**, $A \odot_Q B$ is r–stable, so from Lemma 1**(vi)** we get that $l(A \odot_Q B)$ is l–stable. ∎

Given an LP frame $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$, let us define

$$\mathbf{1'} = l(r(I)) \tag{16}$$

Lemma 19. $\mathbf{1'}$ *is l–stable.*

Proof. Follows from Lemma 1**(ii)** and **(iv)**. ∎

5.2 Complex algebras of LP frames

Definition 14. Let $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ be an LP frame. A ***complex algebra of*** X is a structure $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ with operations defined by (3)–(6), (15) and (16). □

Our aim now is to show that complex algebras of LP frames are LP algebras. To this end, we show that any complex algebra of an LP frame satisfies axioms **(P.1)**–**(P.4)**.

First, we prove that axiom **(P.1)** is satisfied in $L(X)$.

Lemma 20. *Let* $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ *be a complex algebra of an LP frame* X*. Then for every* $A \in L(X)$*,*

- **(i)** $\mathbf{1'} \otimes A = A$
- **(ii)** $A \otimes \mathbf{1'} = A$.

Proof.

(i) Let $A \in L(X)$. We show that $\mathbf{1'} \odot_Q A = r(A)$. Then $l(\mathbf{1'} \odot_Q A) = l(r(A))$, which by the assumption and the definition (15) gives the result.

($\subseteq$) Let $z \in X$ and assume that $z \notin r(A)$. Then there exists $w \in X$ such that **(i.1)** $z \leqslant_2 w$ and **(i.2)** $w \in A$. From **(SP.8)**, there exists $x \in X$ such that **(i.3)** $I(x)$ and **(i.4)** $S(x, w, w)$. Next, from **(MP.4)**, I is $\leqslant_1$–increasing, so by Lemma 4**(ii)**, $I \subseteq l(r(I))$, which together with **(i.3)** implies $x \in l(r(I))$, or equivalently, by the definition (16), **(i.5)** $x \in \mathbf{1'}$. Also, by **(MP.2)**, **(i.1)** and **(i.4)** imply $S(x, w, z)$. Then, in view of **(SP.3)**, there exists $z' \in X$ such that **(i.6)** $z \leqslant_2 z'$ and **(i.7)** $R(x, w, z')$. Furthermore, **(i.7)** implies by **(SP.2)** that there exists $y \in X$ such that **(i.8)** $w \leqslant_1 y$ and **(i.9)** $Q(x, y, z')$. From **(i.6)** and **(i.9)** we get by **(MP.3)** that **(i.10)** $Q(x, y, z)$. Since A is l–stable, by Lemma 1(**i**) it is $\leqslant_1$–increasing. Then **(i.2)** and **(i.8)** imply $y \in A$, so $y \notin r(A)$, which together with **(i.5)** and **(i.10)** gives $z \notin \mathbf{1'} \odot_Q A$ by the definition (13).

($\supseteq$) Let $z \in X$ and assume that $z \notin \mathbf{1'} \odot_Q A$. We show that $z \notin r(A)$.
By assumption, for some $x, y \in X$ we have: **(i.11)** $Q(x, y, z)$, **(i.12)** $x \in \mathbf{1'}$ and **(i.13)** $y \notin r(A)$. From **(i.13)**, there exists $y' \in X$ such that **(i.14)** $y \leqslant_2 y'$ and **(i.15)** $y' \in A$. By **(MP.3)**, from **(i.11)** and **(i.14)** it follows that $Q(x, y', z)$, which by **(SP.4)** implies that there exists $z' \in X$ such that **(i.16)** $z \leqslant_2 z'$ and **(i.17)** $R(x, y', z')$. It suffices to show that $z' \in A$. Then, by **(i.16)** and the definition (1), $z \notin r(A)$.
Suppose that $z' \notin A$. By l–stability of A, this means that $z' \notin l(r(A))$, i.e. there exists $z'' \in X$ such that **(i.18)** $z' \leqslant_1 z''$ and **(i.19)** $z'' \in r(A)$. Applying **(MP.1)**, from **(i.17)** and **(i.18)**, $R(x, y', z'')$, which by **(SP.1)** implies that there exists $x' \in X$ such that **(i.20)** $x \leqslant_1 x'$ and **(i.21)** $S(x', y', z'')$. From **(i.12)**, $x \in \mathbf{1'} = l(r(I))$. Hence, by **(i.20)** and the definition (2), $x' \notin r(I)$, so there exists $x'' \in X$ such that **(i.22)** $x' \leqslant_2 x''$ and **(i.23)** $x'' \in I$. By **(MP.2)**, **(i.21)** and **(i.22)** imply $S(x'', y', z'')$, which by **(SP.3)** gives that there exists $w \in X$ such that **(i.24)** $z'' \leqslant_2 w$ and **(i.25)** $R(x'', y', w)$. Now, applying Lemma 15, **(i.25)**, **(i.23)** and **(i.17)** imply $w \in A$, which together with **(i.24)** gives $z'' \notin r(A)$, which contradicts **(i.20)**.

(ii) Let $A \in L(X)$. As before, we will show that $A \odot_Q \mathbf{1'} = r(A)$. Hence we will have $A \otimes \mathbf{1'} = l(A \odot_Q \mathbf{1'}) = r(l(A)) = A$.

($\subseteq$) Let $z \in X$ and assume that $z \notin r(A)$. By the definition (2) this means that there exists $x \in X$ such that **(ii.1)** $z \leqslant_2 x$ and **(ii.2)** $x \in A$. From **(SP.9)**, there exists $y \in X$ such that **(ii.3)** $y \in I$ and **(ii.4)** $Q(x, y, x)$. From **(MP.4)**, I is $\leqslant_1$–increasing, so by Lemma 4**(ii)**, $I \subseteq l(r(I)) = \mathbf{1'}$. Then from **(ii.3)** it follows that $y \in \mathbf{1'}$, whence **(ii.5)** $y \notin r(\mathbf{1'})$. Next, by **(MP.3)**, **(ii.1)** and **(ii.4)** imply **(ii.6)** $Q(x, y, z)$. Therefore, there exist $x, y \in X$ such that **(ii.2)**, **(ii.5)** and **(ii.6)** hold, which by the definition (13) means that $z \notin A \odot_Q \mathbf{1'}$.

($\supseteq$) We have to show that $r(A) \subseteq A \odot_Q \mathbf{1'}$. Assume that $z \notin A \odot_Q \mathbf{1'}$. By the definition (13), there exist $x, y \in X$ such that **(ii.7)** $Q(x, y, z)$, **(ii.8)** $x \in A$ and **(ii.9)** $y \notin r(\mathbf{1'})$. From **(ii.9)**, there exists $y' \in X$ such that **(ii.10)** $y \leqslant_2 y'$ and **(ii.11)** $y' \in \mathbf{1'}$. By **(MP.3)**, **(ii.7)** and **(ii.10)** imply $Q(x, y', z)$, from which, by **(SP.4)**, it follows that there exists $z' \in X$ such that **(ii.12)** $z \leqslant_2 z'$ and $R(x, y', z')$. Proceeding in the similar way as in the proof of ($\supseteq$) in **(i)**, we can show that $z' \in A$, which together with **(ii.12)** implies $z \notin r(A)$. ∎

The following lemma states that **(P.2)** is satisfied in $L(X)$.

Lemma 21. *Let $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ be the complex algebra of an LP frame X. Then for all $A, B, C \in L(X)$, the following equality holds:*

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C.$$

Proof. Let $A, B, C \subseteq X$ be l–stable sets. In view of Lemma 16 it suffices to show that $A \odot_Q (B \otimes C) = (A \otimes B) \odot_S C$.

($\subseteq$) Let $z \in X$ and assume that $z \notin (A \otimes B) \odot_S C$. By the definition (14), this means that there exist $x, y \in X$ such that **(1)** $S(x, y, z)$, **(2)** $y \in C$ and **(3)** $x \notin r(A \otimes B)$. But $r(A \otimes B) = r(l(A \odot_Q B))$ by the definition (15). Next, by Lemma 16, $r(l(A \odot_Q B)) = r(l(A \odot_S B))$. Moreover, $A \odot_S B$ is r–stable by Lemma 17**(ii)**, so $r(l(A \odot_S B)) = A \odot_S B$. Then we have $r(A \otimes B) = A \odot_S B$. Hence, from **(3)** we get $x \notin A \odot_S B$, which means that there exist $u, v \in X$ such that **(4)** $S(u, v, x)$, **(5)** $v \in B$ and **(6)** $u \notin r(A)$. From **(6)**, there exists $u' \in X$ such that **(7)** $u \leqslant_2 u'$ and **(8)** $u' \in A$. By the monotonicity condition **(MP.2)**, **(4)** and **(7)** imply $S(u', v, x)$, which by **(SP.3)** gives that there exists $x' \in X$ such that **(9)** $x \leqslant_2 x'$ and **(10)** $R(u', v, x')$. Applying again **(MP.2)**, from **(1)** and **(9)** we get that $S(x', y, z)$. Therefore, there exists $x' \in X$ such that $R(u'v, x')$ and $S(x', y, z)$. By **(SP.6)**, this implies that there exists $w \in X$ such that **(11)** $R(v, y, w)$ and **(12)** $Q(u', w, z)$. Next, by **(SP.1)**, **(11)** implies that there exists $v' \in X$ such that **(13)** $v \leqslant_1 v'$ and **(14)** $S(v', y, w)$. Since B is l–stable, it is $\leqslant_1$–increasing. Then **(5)** and **(13)** imply $v' \in B$, so **(15)** $v' \notin r(B)$. We have then obtained that there exist $y, v' \in X$ such that **(2)**, **(14)** and **(15)** hold, which by the definition (14) means that **(16)** $w \notin B \odot_S C$. From Lemma 17, $B \odot_S C$ is r–stable, so $B \odot_S C = r(l(B \odot_S C)) = r(l(B \odot_Q C)) = r(B \otimes C)$

by Lemma 16. Whence **(16)** implies **(17)** $w \notin r(B \otimes C)$. Then we finally get that there exist $u', w \in X$ such that **(8)**, **(12)** and **(17)** hold. By the definition (13), this means that $z \notin A \odot_Q (B \otimes C)$.

Proceeding in the similar way, and using **(SP.5)**, **(⊇)** can be proved. ■

Finally, we show that axioms **(P.3)** and **(P.4)** are satisfied in complex algebras of LP frames.

Lemma 22. *Let* $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ *be the complex algebra of an LP frame* X*. Then for all* $A, B, C \in L(X)$*,*

- **(i)** $A \otimes (B \sqcup C) = (A \otimes B) \sqcup (A \otimes C)$
- **(ii)** $(B \sqcup C) \otimes A = (B \otimes A) \sqcup (C \otimes A)$.

Proof.

(i) Let $A, B, C \subseteq X$ be l–stable sets. First we show that **(i.1)** $A \odot_Q (B \sqcup C) = (A \odot_Q B) \cap (A \odot_Q C)$.
For every $x \in X$ the following equivalences hold:

$$\begin{aligned}
&x \in (A \odot_Q B) \cap (A \odot_Q C) \\
&\quad \text{iff} \;\; x \in A \odot_Q B \;\&\; x \in A \odot_Q C \\
&\quad \text{iff} \;\; [\forall y, z \in X\ (Q(y,z,x) \;\&\; y \in A \Rightarrow z \in r(B))] \\
&\qquad\quad \& \;\; [\forall y, z \in X\ (Q(y,z,x) \;\&\; y \in A \Rightarrow z \in r(C))] \\
&\quad \text{iff} \;\; \forall y, z \in X\ (Q(y,z,x) \;\&\; y \in A \Rightarrow z \in r(B) \cap r(C)).
\end{aligned}$$

By Lemma 1**(vii)**, $r(B) \cap r(C)$ is r–stable, so $r(B) \cap r(C) = r(l(r(B) \cap r(C)))$. Hence, by the definition (4) we get

$$\begin{aligned}
&x \in (A \odot_Q B) \cap (A \odot_Q C) \\
&\quad \text{iff} \;\; \forall y, z, \in X\ (Q(y,z,x) \;\&\; y \in A \Rightarrow z \in r(l(r(B) \cap r(C)))) \\
&\quad \text{iff} \;\; x \in A \odot_Q l(r(B) \cap r(C)) \\
&\quad \text{iff} \;\; x \in A \odot_Q (B \sqcup C).
\end{aligned}$$

So **(i.1)** holds. Hence **(i.2)** $l(A \odot_Q (B \sqcup C)) = l((A \odot_Q B) \cap (A \odot_Q C))$. Note that **(i.3)** $l(A \odot_Q (B \sqcup C)) = A \otimes (B \sqcup C)$. Next, by Lemma 17**(ii)**, $A \odot_Q B$ is r–stable, so $r(l(A \odot_Q B)) = A \odot_Q B$. Using again the definition (4), we get

$$\begin{aligned}
l((A \odot_Q B) \cap (A \odot_Q C)) &= l(r(l(A \odot_Q B)) \cap r(l(A \odot_Q C))) \\
&= l(A \odot_Q B) \sqcup l(A \odot_Q C) \\
&= (A \otimes B) \sqcup (A \otimes C).
\end{aligned}$$

Hence, by **(i.2)** and **(i.3)** we get the required result.

(ii) can be proved in the similar way. ■

From Lemmas 20, 21 and 22 we get:

Theorem 6. *The complex algebra of an LP frame is an LP algebra.* ■

5.3 Canonical frames of LP algebras

Let $(W, \wedge, \vee, 0, 1, \odot, 1')$ be an LP algebra. As before, by a *filter* (resp. *ideal*) of W we mean a filter (resp. ideal) of the underlying lattice $(W, \wedge, \vee, 0, 1)$. We will write $X(W)$ to denote the family of all maximal filter–ideal pairs of the lattice reduct of W.

Let us define the following ternary relations on $X(W)$: for all $x, y, z \in X(W)$,

$$R^\star(x,y,z) \quad \text{iff} \quad (\forall a, b \in W)\ a \in x_1 \ \&\ b \in y_1 \Rightarrow a \odot b \in z_1 \tag{17}$$

$$S^\star(x,y,z) \quad \text{iff} \quad (\forall a, b \in W)\ a \odot b \in z_2 \ \&\ b \in y_1 \Rightarrow a \in x_2 \tag{18}$$

$$Q^\star(x,y,z) \quad \text{iff} \quad (\forall a, b \in W)\ a \odot b \in z_2 \ \&\ a \in x_1 \Rightarrow b \in y_2 \tag{19}$$

Moreover, let

$$I^\star = \{x \in X(W) : 1' \in x_1\}. \tag{20}$$

Definition 15. Let an LP algebra $(W, \wedge, \vee, 0, 1, \odot, 1')$ be given. The structure $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ is called a ***canonical frame of*** W. □

The following two lemmas can be proved as in [1].

Lemma 23. *Let $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ be the canonical frame of an LP algebra. Then $R^\star$, $S^\star$, $Q^\star$ and $I^\star$ satisfy monotonicity conditions* **(MP.1)**–**(MP.4)** *of Definition 13.* ■

Lemma 24. *Let $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ be the canonical frame of an LP algebra. Then $R^\star$, $S^\star$, $Q^\star$ and $I^\star$ satisfy stability conditions* **(SP.1)**–**(SP.9)** *of Definition 13.* ■

Lemmas 23 and 24 imply the following theorem:

Theorem 7. *The canonical frame of an LP algebra is an LP frame.* ■

5.4 Relational representation for LP algebras

Let $(W, \wedge, \vee, 0, 1, \odot, 1')$ be an LP algebra, $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ be its canonical frame and let $(L(X(W)), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ be the complex algebra of $X(W)$. Let the mapping $h : W \to 2^{X(W)}$ be defined as in (9), i.e. for every $a \in W$,

$$h(a) = \{x \in X(W) : a \in x_1\}.$$

Our aim is to show that W is isomorphic to a subalgebra of $L(X(W))$.

To begin, we introduce the following auxiliary notation: Given an LP algebra, for any $A, B \subseteq W$ denote

$$A \odot B = \{a \odot b : a \in A \ \& \ b \in B\}.$$

First, note the following

Lemma 25. *Let an LP algebra $(W, \wedge, \vee, 0, 1, \odot, 1')$ be given and let F and I be a filter and an ideal of W, respectively. Then*

$$U = \{a \in W : (\{a\} \odot F) \cap I \neq \emptyset\}$$
$$V = \{a \in W : (F \odot \{a\}) \cap I \neq \emptyset\}$$

are ideals of W.

Proof. We show that U is an ideal of W. Let **(1)** $a \in U$ and **(2)** $b \leqslant a$. From the definition of U, **(1)** implies that there is $c \in F$ such that **(3)** $a \odot c \in I$. By Lemma 14**(i)**, **(2)** implies **(4)** $b \odot c \leqslant a \odot c$. Since I is a ideal, **(3)** and **(4)** give **(5)** $b \odot c \in I$. So for some $c \in F$, **(5)** holds, which gives $b \in U$.

Let $a, b \in U$. We shall show that $a \vee b \in U$. By assumption, there exist $c, d \in F$ such that **(6)** $a \odot c \in I$ and **(7)** $b \odot c \in I$. Since $c \wedge d \leqslant c$, by Lemma 14**(i)** we have that $a \odot (c \wedge d) \leqslant a \odot c$ and $b \odot (c \wedge d) \leqslant b \odot c$, so by **(6)** and **(7)** we get $(a \odot (c \wedge d) \in I$ and $(b \odot (c \wedge d) \in I$. Since I is an ideal, $(a \odot (c \wedge d)) \vee (b \odot (c \wedge d)) \in I$. From axiom **(P.4)**, $(a \odot (c \wedge d)) \vee (b \odot (c \wedge d)) = (a \vee b) \odot (c \wedge d)$, so **(8)** $(a \vee b) \odot (c \wedge d) \in I$. Since F is a filter, $c \wedge d \in F$. So we have shown that for some $c' = c \wedge d \in F$, $(a \vee b) \odot c' \in I$, which by the definition of U implies that $a \vee b \in U$.

Proceeding in the similar way we can show that V is an ideal. ∎

Theorem 8 (Representation theorem for LP algebras). *Every LP algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. We have to show that

- **(i)** $h(1') = \mathbf{1'}$
- **(ii)** $h(a \odot b) = h(a) \otimes h(b)$.

(i) Note that by (20), $I^\star = h(1')$, so from Theorem 1**(ii)**, it is an l–stable set. Also, $\mathbf{1'} = l(r(I^\star))$ by (16). Hence $\mathbf{1'} = I^\star$, i.e. $h(1') = \mathbf{1'}$.

(ii) **($\subseteq$)** Let $a, b \in W$, $z \in X(W)$ and assume that $z \in h(a \odot b)$. Then it holds **(ii.1)** $a \odot b \in z_1$. We have to show that $z \in h(a) \otimes h(b)$. By the definition (15) and Lemma 16, this means that for any $w \in X(W)$, if $z \preccurlyeq_1 w$, then $w \notin h(a) \odot_S h(b)$. Assume that $z \preccurlyeq_1 w$, i.e. **(ii.2)** $z_1 \subseteq w_1$. We will show that **(ii.3)** $w \notin h(a) \odot_S h(b)$.

Let $[a)$ be the filter generated by a, i.e. $[a) = \{e \in W : a \leqslant e\}$. Define

$$U = \{c \in W : ([a) \odot \{c\}) \cap w_2 \neq \emptyset\}.$$

By Lemma 25, U is an ideal. We show that **(ii.4)** $b \notin U$. Suppose that $b \in U$. Then there exists $e \in [a)$ such that **(ii.5)** $e \odot b \in w_2$. Since $e \in [a)$, $a \leqslant e$, so $a \vee e = e$. Hence **(ii.6)** $(a \vee e) \odot b = e \odot b$. By axiom **(P.4)**, $(a \vee e) \odot b = (a \odot b) \vee (e \odot b)$. So we have $(a \odot b) \vee (e \odot b) = e \odot b$, whence $a \odot b \leqslant e \odot b$. Then, since w_2 is an ideal, by **(ii.5)** we get $a \odot b \in w_2$, so $a \odot b \notin w_1$, which by **(ii.2)** gives $a \odot b \notin z_1$ – a contradiction with **(ii.1)**.
Then $([b), U) \in FIP(W)$. Let (y_1, y_2) be its extension to the maximal filter–ideal pair. Hence **(ii.6)** $[b) \subseteq y_1$ and **(ii.7)** $U \subseteq y_2$. From **(ii.6)**, $b \in y_1$, so **(ii.8)** $y \in h(b)$.

Next, let us consider the set

$$V = \{c \in W : (\{c\} \odot y_1) \cap w_2 \neq \emptyset\}.$$

By Lemma 25, V is an ideal. We show that **(ii.9)** $a \notin V$. Suppose that $a \in V$. Then there exists $e \in L$ such that **(ii.10)** $e \in y_1$ and **(ii.11)** $a \odot e \in w_2$. By the definition of U, **(ii.11)** means that $e \in U$, so by **(ii.7)**, $e \in y_2$. Whence $e \notin y_1$, which contradicts **(ii.10)**. So **(ii.9)** was proved. Then $([a), V) \in FIP(W)$. Let (x_1, x_2) be its extension to the maximal filter–ideal pair. Then we have **(ii.12)** $[a) \subseteq x_1$ and **(ii.13)** $V \subseteq x_2$. From **(ii.13)**, $a \in x_1$, so $x \in h(a)$, which implies **(ii.14)** $x \notin r(h(a))$.
Finally, we show that **(ii.15)** $S^\star(x, y, w)$. By the definition (18) of $S^\star$, this means that for all $c, d \in W$, $c \notin x_2$ and $d \in y_1$ imply $c \odot d \notin w_2$. Let $c, d \in W$ be such that $c \notin x_2$ and $d \in y_1$. It suffices to show that $c \odot d \notin e_2$. Indeed, $c \notin x_2$ implies by **(ii.13)** that $c \notin V$. From the definition of V, this gives that for any $e \in y_1$, $c \odot e \notin w_2$, so in particular $c \odot d \notin w_2$. Therefore, for some $x, y \in X(W)$, **(ii.8)**, **(ii.14)** and **(ii.15)** hold – by the definition (14) of $\odot_S$ **(ii.3)** follows.

($\supseteq$) Let $a, b \in L$ and let $z \in X(W)$. Assume that $z \in h(a) \otimes h(b)$. By the definition (15) this is equivalent to **(ii.16)** $z \in l(h(a) \odot_Q h(b))$. Let $y \in X(W)$ be such that **(ii.17)** $z_1 \subseteq y_1$. Then from **(ii.16)**, $y \notin h(a) \odot_Q h(b)$, which means that there exist $x, w \in X(W)$ such that **(ii.18)** $Q^\star(x, w, y)$, **(ii.19)** $x \in h(a)$ and **(ii.20)** $w \notin r(h(b))$. From **(ii.19)**, $a \in x_1$. Next, by Theorem 1**(i)**, **(ii.20)** implies **(ii.21)** $b \notin w_2$. By the definition (19), **(ii.18)** gives that for all $a, b \in W$, $a \odot b \in y_2$ and $a \in x_1$ imply $b \in w_2$. Hence, from **(ii.19)** and **(ii.21)**, $a \odot b \notin y_2$. Applying again Theorem 1**(i)**, we obtain $y \notin r(h(a \odot b))$, which together with **(ii.17)** gives $z \in l(r(h(a \odot b)))$. Since by Theorem 1**(ii)**, $h(a \odot b)$ is l–stable, we finally get $z \in h(a \odot b)$. ∎

6 LCP algebras

LCP algebras are meant to be relation algebras based on arbitrary bounded lattices. Their axioms consist of the axioms **(C.1)** and **(C.2)** of converse, the axioms **(P.1)**,...,**(P.4)** of product and an additional axiom which tells us how converse and product are related with each other. In the axiomatization of classical relation algebras, this is done by postulating that converse distributes over composition and also by the axiom

$$a \odot -(a^{\smile} \odot -b) \leqslant b, \tag{21}$$

where $-x$ is the Boolean complement of x. It is well known that (21) is equivalent to de Morgan's *Theorem K*, one form of which states that

$$a \odot b \leqslant -c \text{ iff } a^{\smile} \odot c \leqslant -b \text{ iff } c \odot b^{\smile} \leqslant -a. \tag{22}$$

In our present setting, we do not have complementation as a distinguished operation, and thus, we cannot use (21). It may be argued that one could use the complement free version of (22), namely,

$$(a \odot b) \wedge c = 0 \text{ iff } (a^{\smile} \odot c) \wedge b = 0 \text{ iff } (c \odot b^{\smile}) \wedge a = 0. \tag{23}$$

However, it is not quite clear whether this is useful because of the following: Suppose that W is an LCP algebra (formally defined below), and let $m \notin W$. Set $W' = W \cup \{m\}$, and extend ordering and the operations of W over W' by

$$m \leqslant x, \quad m^{\smile} = m, \quad m \odot x = x \odot m = m$$

for all $x \in W'$. In other words, we are adding a new smallest element to W.

Lemma 26.

1. *W' is an LCP algebra which satisfies* (23)
2. *If $\tau = \sigma$ is an equation of LCP algebras not containing* 0*, then*

$$W \models \tau = \sigma \text{ iff } W' \models \tau = \sigma.$$

Proof. The idea is to show that the equational classes generated by W and W' are the same when 0 is omitted the signature. Suppose that $Eq(W)$ and $Eq(W')$ are these classes. It suffices to show that $W \in Eq(W')$ and $W' \in Eq(W)$. The first claim follows from the fact that W is a subalgebra of W'. The second claim can be seen as follows. Let $f : W' \to W \times W$ defined by

$$f(x) = \begin{cases} \langle x, 1\rangle, & \text{if } x \neq m, \\ \langle 0, 0\rangle, & \text{if } x = m. \end{cases}$$

Then, f is an injective homomorphism, showing that $W' \in ISP(W)$. $\square$

It may also be worthy to note that (21) is equivalent to the statement

$$-(a^{\smile} \odot -b) \text{ is the largest } x \text{ with } a \odot x \leqslant b,$$

i.e. $-(a^{\smile} \odot - b)$ is the (right) residuum of $\odot$. Hence, a more coherent way would be to introduce residuals as e.g. in [22] as distinguished operators.

Let us formally define LCP algebras:

Definition 16. An ***LCP algebra*** is a system $(W, \wedge, \vee, 0, 1, {}^{\smile}, \odot, 1')$ such that $(W, \wedge, \vee, 0, 1, {}^{\smile})$ is an LC algebra, $(W, \wedge, \vee, 0, 1, \odot, 1')$ is an LP algebra and for all $a, b \in W$ the following holds:

(CP) $(a \odot b)^{\smile} = b^{\smile} \odot a^{\smile}$. $\square$

Note that

Lemma 27. *For every LCP algebra $(W, \wedge, \vee, 0, 1, {}^{\smile}, \odot, 1')$, $1'$ is an equivalence element.*

Proof. We have to show that $1'$ is transitive and symmetric. Transitivity follows from $1' \odot 1' = 1'$, and symmetry can be shown as follows:

$$\begin{aligned}
1'^{\smile} &= 1'^{\smile} \odot 1', && \text{by } \textbf{(P.1)}\\
&= (1'^{\smile} \odot 1')^{\smile\smile}, && \text{by } \textbf{(C.1)}\\
&= (1'^{\smile} \odot 1')^{\smile}, && \text{by } \textbf{(CP)} \text{ and } \textbf{(C.1)}\\
&= 1'^{\smile\smile}, && \text{by } \textbf{(P.1)}\\
&= 1', && \text{by } \textbf{(C.1)}. \quad \square
\end{aligned}$$

Example 1. Let us consider a system $(W, \wedge, \vee, 0, 1, {}^{\smile}, 1')$ such that $W = \{0, a, b, c, d, e, 1\}$, $1' = b$ and the operations ${}^{\smile}$ and $\odot$ are given in Tables 1 and 2 below, respectively.

x	$x^{\smile}$
0	0
a	a
b	b
c	d
d	c
e	e
1	1

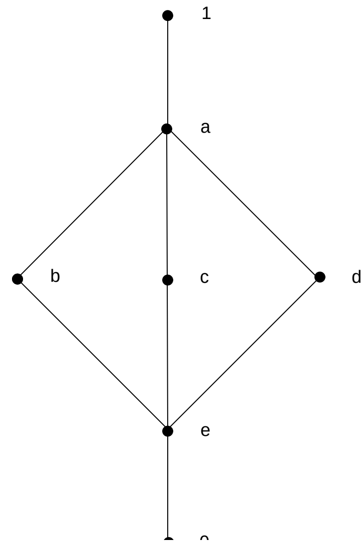
Table 1. Converse ${}^{\smile}$ of W

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	c	d	e	1
b	0	a	b	c	d	e	1
c	0	c	c	c	e	e	c
d	0	d	d	0	d	0	d
e	0	e	e	0	e	0	e
1	0	1	1	c	d	e	1

Table 2. Product $\odot$ of W

By straightforward verification it is easy to check that W is an LCP algebra. Note that the underlying lattice is not distributive and product of W is non–commutative ($c \odot e = e \neq 0 = e \odot c$). $\square$

Fig. 1. An example



As before, we start our discussion on LCP algebras by defining *LCP frames*.

Definition 17. An ***LCP frame*** is a system $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ such that $(X, \leqslant_1, \leqslant_2, C)$ is an LC frame, $(X, \leqslant_1, \leqslant_2, R, S, Q, I)$ is an LP frame and moreover, for all $x, y, z \in X$,

(SCP) $Q(x, y, z) = S(C(y), C(x), C(z))$. $\square$

Definition 18. Let $(X, \preccurlyeq_1, \preccurlyeq_2, C, R, S, Q, I)$ be an LCP frame. A ***complex algebra of*** X is a structure $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, ^{\curlyvee}, \otimes, \mathbf{1'})$ with operations defined by (3)–(6), (11), (15) and (16). $\square$

Theorem 9. *The complex algebra of an LCP frame is an LCP algebra.*

Proof. Let $(X, \preccurlyeq_1, \preccurlyeq_2, C, R, S, Q, I)$ be an LCP frame and let $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, ^{\curlyvee}, \otimes, \mathbf{1'})$ be its complex algebra. From Theorems 4 and 6 it follows that $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, ^{\curlyvee})$ is an LC algebra and $(L(X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1}, \otimes, \mathbf{1'})$ is an LP algebra. It suffices to show that **(CP)** holds in $L(X)$, i.e. $(A \otimes B)^{\curlyvee} = B^{\curlyvee} \otimes A^{\curlyvee}$ for all l–stable sets $A, B \subseteq X$.

(⊆) Let $z \in (A \otimes B)^{\curlyvee}$. This is means that $C(z) \in A \otimes B$, or equivalently, **(1)** $C(z) \in l(A \odot_Q B)$. Let $z' \in X$ be such that **(2)** $C(z) \leqslant_1 z'$. By **(MC.1)** and **(SC)** this is equivalent to **(3)** $z \leqslant_1 C(z')$. From **(1)** and **(2)** we get $z' \notin A \odot_Q B$. By the definition (13) this means that there exist $x, y \in X$ such that **(4)** $Q(x, y, z')$, **(5)** $x \in A$ and **(6)** $y \notin r(B)$. From **(6)**, there exists $y' \in X$ such that

(7) $y \leqslant_2 y'$ and **(8)** $y' \in B$. Next, **(5)** implies **(9)** $C(x) \in A^{\curlyvee}$. Similarly, from **(8)** it follows $C(y') \in B^{\curlyvee}$, so **(10)** $C(y') \notin r(B^{\curlyvee})$. By **(MP.3)**, **(4)** and **(7)** imply $Q(x, y', z')$, which by **(SCP)** is equivalent to **(11)** $S(C(y'), C(x), C(z'))$. Hence, by **(9)** and **(10)** we get that $C(z') \notin B^{\curlyvee} \odot_S A^{\curlyvee}$. Since A and B are l–stable, by Lemma 11**(iii)** $A^{\curlyvee}$ and $B^{\curlyvee}$ are also l–stable. Hence, from Lemma 16, $B^{\curlyvee} \odot_S A^{\curlyvee} = B^{\curlyvee} \odot_Q A^{\curlyvee}$ which together with **(3)** gives $z \in l(B^{\curlyvee} \odot_Q A^{\curlyvee})$, that is $z \in B^{\curlyvee} \otimes A^{\curlyvee}$.

The proof of ($\supseteq$) is similar. ■

Let $(W, \wedge, \vee, 0, 1, ^{\smile}, \odot, 1')$ be an LCP algebra. As before, by a *filter* (*ideal*) of W we mean a filter (ideal) of the underlying lattice $(W, \wedge, \vee, 0, 1)$. We will write $X(W)$ to denote the family of all maximal filter–ideal pairs of the lattice reduct of W.

Definition 19. Let $(W, \wedge, \vee, 0, 1, ^{\smile}, \odot, 1')$ be an LCP algebra. A ***canonical frame of*** W is a structure $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, S^\star, Q^\star, I^\star)$ such that $\preccurlyeq_1$ and $\preccurlyeq_2$ are defined by (7) and (8), respectively, and relations $C^\star$, $R^\star$, $S^\star$, $Q^\star$ and $I^\star$ are defined by (12) and (17)–(20), respectively. □

Theorem 10. *The canonical frame of an LCP algebra is an LCP frame.*

Proof. Let $(X, \wedge, \vee, 0, 1, ^{\smile}, \odot, 1')$ be an LCP algebra and let $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, S^\star, Q^\star, I^\star)$ be its canonical frame. By Theorem 4, $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star)$ is an LC frame. Next, by Theorem 7, $(X(W), \preccurlyeq_1, \preccurlyeq_2, R^\star, S^\star, Q^\star, I^\star)$ is an LP frame. It suffices to show that **(SCP)** holds.
Let $x, y, z \in X(W)$. We have the following equivalences:

$S^\star(C^\star(y), C^\star(x), C^\star(z))$

iff	$(\forall a, b \in W)\ a \odot b \in z_2^{\smile}\ \&\ b \in x_1^{\smile} \Rightarrow a \in y_2^{\smile}$	by (18)
iff	$(\forall a, b \in W)\ (a \odot b)^{\smile} \in z_2\ \&\ b^{\smile} \in x_1 \Rightarrow a^{\smile} \in y_2$	by (10)
iff	$(\forall a, b \in W)\ b^{\smile} \odot a^{\smile} \in z_2\ \&\ b^{\smile} \in x_1 \Rightarrow a^{\smile} \in y_2$	by axiom **(CP)**
iff	$Q^\star(x, y, z)$	by (19). ■

We conclude the paper by showing the representability of LCP algebras.

Theorem 11. *Every LCP algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. Follows from Theorems 5, 8, 9 and 10. ■

7 Conclusions

In this paper we have studied not necessarily distributive lattices with the operators which are the abstract counterparts to the converse and composition of binary relations. On the algebraic side, we have presented relational representation theorems for these classes of algebras. The representation theorems are obtained by a suitable extensions of the Urquhart representation theorem for lattices [25]. However, here we stress the relational aspect of representability, and we omit the topological aspect. On the logical side, with every class of algebras studied in the paper we associate an appropriate class of frames. These frames constitute a basis of a Kripke-style semantics for the logics whose algebraic semantics is determined by the classes of algebras presented in the paper. The representation theorems would enable us to prove completeness of the logics. For a detailed elaboration of the respective relational logics one can follow the developments in [1] and [23].

The further work is planned on the class of relation algebras based on double residuated lattices presented in [22]. The signature of such algebras extends the signature of LCP algebras with the residuation operators determined by the product, and with the sum and its dual residua. In these algebras complement operations are definable in terms of residua. Therefore the axioms of this class of algebras should include some counterparts to the De Morgan theorem K.

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