# BROCK UNIVERSITY MATHEMATICS MODULES

# Module D1.5: Instantaneous Rates of Change

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### WWW

- What it is: The average rate of change of a function y = f(x) on an interval [a, b] is the slope of the secant line joining the points P(a, f(a)) and Q(b, f(b)). Keeping the point P fixed and allowing the point Q to move ever closer to the point P, the resulting sequence of values for the average rates of change approaches the instantaneous rate of change of the function f at the point P. The graphical interpretation of instantaneous rate of change is the slope of a tangent line to a curve.
- Why you need it: The relation of an instantaneous rate of change to a sequence of average rates of change is a foundational idea of calculus. Instantaneous rates of change appear in all applications of calculus, and are fundamental in scientific descriptions of nature.
- When to use it: The instantaneous rate of change of your car's position is indicated by the value on its speedometer. Use the instantaneous rate of change whenever you need to know the precise rate of change of a function at a specific value of its domain.

#### PREREQUISITES

Before you tackle this module, make sure you have completed these modules:

Introduction to Rates of Change 12D1.1, 12D1.2, 12D1.3, Average Rates of Change 12D1.4

#### WARMUP

Before you tackle this module, make sure you can solve the following exercises. If you have difficulties, please review the appropriate prerequisite modules.

(Answers below.)

1. Calculate the average rate of change on the interval [-1,3] and draw the secant line.





#### Introduction

In previous modules you have learned that the slope of a secant line that joins two points P(a, f(a)) and Q(b, f(b)) on a graph represents the average rate of change of the function f on the interval [a, b]. For example, suppose f(x) represents the position of a car at time x; then the slope of the secant line joining P and Q represents the average speed<sup>1</sup> of the car for the time interval [a, b]. However, the actual speed at any moment, called the *instantaneous speed*, is indicated on the car's speedometer.

How can we calculate the instantaneous speed of a car if we have a formula for its position function? This is one of the main tasks of calculus, and in this module we describe the basic idea behind such a calculation, and its graphical interpretation. By studying calculus, you will understand the concept in much greater detail, and be able to perform such calculations for a wide variety of functions.<sup>2</sup>

The basic idea is this: If you make the time interval smaller and smaller, so that the point Q "approaches" the point P along the graph of f, then the slope of the secant line joining P and Q approximates the instantaneous rate of change of the function f at the point P more and more closely.<sup>3</sup> In other words, as the time interval is made smaller and smaller, the average rate of change of the function approaches the instantaneous rate of change.

In this module we'll perform some numerical calculations to explore this idea.

<sup>&</sup>lt;sup>1</sup>There are subtleties here that we are intentionally overlooking; for a deeper understanding, study the distinction between speed and velocity. Strictly speaking, the slope of the position-time graph represents the velocity of the car, not its speed.

<sup>&</sup>lt;sup>2</sup>If you intend to pursue this, some key words to search for are "limit" and "derivative."

<sup>&</sup>lt;sup>3</sup>Once again, we are purposely overlooking important issues, because this module is exploratory. For instance, the procedure described here doesn't always work. Once you are familiar with the procedure, you might like to think about the situations when it will work, and the situations when it does not work.

#### FOCUS QUESTION

To help you understand an important aspect of this lesson, focus your attention on this question, which will be answered towards the end of the lesson.

A bacterial culture grows so that the number N(t) of cells at time t hours after an experiment begins is  $N(t) = 3t^2 + 100$ . Estimate the rate at which the bacterial colony grows 4 hours after the experiment begins.

To understand the relationship between average rates of change and instantaneous rates of change, let's return to Warmup Question 2. The table of values shows the average rate of change for smaller and smaller intervals of x that include x = 0 as the left endpoint of the interval. What would it look like if we graphed each of the secant lines?



Notice that for each secant line, the two endpoints become closer and closer together as the interval gets smaller. This is exactly what we want in order to estimate the instantaneous rate of change of the cosine function at x = 0. This is because the secant lines become better and better approximations to the graph of  $y = \cos x$  as the interval becomes smaller. That is, if you were to walk down the cosine curve or walk down one of the secant lines, then near x = 0 the slopes of the secant lines become closer and closer to the slope of the cosine curve as the interval over which the slope of the secant line is calculated becomes smaller and smaller.

Here is an example to illustrate the fact that slopes of secant lines (which represent average rates of change) become better and better approximations to the actual instantanteous rate of change of a function at a point when the interval over which the slope of the secant line is calculated becomes smaller and smaller.

#### EXAMPLE 1

Use secant lines over smaller and smaller intervals near x = 1 to estimate the instantaneous rate of change of the function  $f(x) = x^2$  at x = 1.

#### SOLUTION

Let's select a point nearby to P(1,1), which is the point at which we wish to estimate the slope of the graph. How about Q(2,4), shall we? Then the slope of the secant line PQ is

slope of PQ = 
$$\frac{\text{rise}}{\text{run}}$$
  
=  $\frac{y_2 - y_1}{x_2 - x_1}$   
=  $\frac{4 - 1}{2 - 1}$   
=  $\frac{3}{1}$   
= 3

So, the slope of the secant line PQ is 3. This means that the actual slope of the curve at x = 1 is "approximately 3." To improve the approximation, we can move the point Q closer to the point P; let's say we move the point Q until it reaches R(1.5, 2.25). Repeating the calculation for the slope of the secant line PR, we get,

slope of PR = 
$$\frac{\text{rise}}{\text{run}}$$
  
=  $\frac{y_2 - y_1}{x_2 - x_1}$   
=  $\frac{2.25 - 1}{1.5 - 1}$   
=  $\frac{1.25}{0.5}$   
= 2.5

The slope of the secant line PR is 2.5, which means that the actual slope of the curve at x = 1 is "approximately 2.5." See Figure 1.





Figure 1: The secant line PQ approximates the graph of  $f(x) = x^2$  near x = 1.

Figure 2: The secant line PR is a better approximation to the graph near x = 1 than PQ. The actual slope of the graph at P is less than either of the slopes of the secant lines.

However, there is a problem with these kinds of approximations, because we have no idea how accurate they are. In everyday language, there is an implicit sense of the accuracy of a reported approximation, but in mathematics and science there is no implicit sense; we have to be explicit in stating how accurate the approximation is. For example, I might say to you, "It is about 10 km from Toronto to Vancouver."<sup>a</sup> You would be perfectly justified in replying that this is not true, that the distance is far greater than 10 km. I might try to weasel my way out of this by saying, "Yes, but what I said was only an *approximation*." Nevertheless, you would have every right to consider me silly, because there is an unspoken understanding that "about 10 km" might mean 7 km, or 8 km, or 12 km, or 14 km, but it certainly does not mean 30 km, or 40 km, or 100 km, and it certainly does not mean several thousand kilometres.

The point is that we have no idea how accurate the approximations 3 or 2.5 are for the true slope of the graph of  $f(x) = x^2$  at x = 1. What we need is some sort of procedure that will allow us some sense for how accurate these approximations are. If, in addition, we could come up with a procedure for systematically improving the approximations, this would be great. We will now introduce the basic idea behind such a procedure, which is one of the most important fundamental ideas in mathematical analysis. (You will learn about this in more detail when you take calculus; the key word is "limit.")

<sup>&</sup>lt;sup>a</sup>The actual road distance from Toronto to Vancouver (through Canada; the distance via the U.S. is less) is 4500 km, and the actual distance by airplane is 3350 km. At least these are the *approximate* distances, if you know what I mean! (Besides the usual uncertainties of making measurements, the distance also depends on the specific route.)

From the graphs in Figures 1 and 2, you can see that the slope of the curve at x = 1 is less than the slope of the secant line PQ. (It is steeper walking up either of the secant lines than it is walking up the curved graph just at x = 1, right?) We calculated that the slope of the secant line PQ is 3. This means that the true slope of the curve at x = 1 is less than 3. How much less than 3 we can't say, and that is the problem. The same argument applied to secant line PRleads us to conclude that the true slope of the curve at x = 1 is less than 2.5. How much less? We can't say.

Now consider Figure 3.



Figure 3: The secant line PS approximates the graph of  $f(x) = x^2$  near x = 1; the secant line PT is a better approximation near x = 1. The actual slope of the graph at P is greater than either of the slopes of the secant lines.

Apparently it is "easier" to walk from left to right up the secant lines than it is to walk up the curve near x = 1. That is, the actual slope of the curve at x = 1 is greater than the slopes of the secant lines PS and PT. Is it also clear to you that the secant line PT is a better approximation to the curve at x = 1 than the secant line PS? If you see this, then do you also see that as the point T is moved closer and closer to the point P, then the secant line becomes a better and better approximation to the curve at P?

Let's calculate the slope of the secant line PS:

slope of PS = 
$$\frac{\text{rnse}}{\text{run}}$$
  
=  $\frac{y_2 - y_1}{x_2 - x_1}$   
=  $\frac{1 - 0.25}{1 - (-0.5)}$   
=  $\frac{0.75}{1.5}$   
= 0.5

This means that the actual slope of the curve at x = 1 is approximately 0.5; in fact it's greater than 0.5, as we argued above.

The slope of the secant line PT is:

slope of PS = 
$$\frac{\text{rise}}{\text{run}}$$
  
=  $\frac{y_2 - y_1}{x_2 - x_1}$   
=  $\frac{1 - 0}{1 - 0}$   
=  $\frac{1}{1}$   
= 1

This means that the true slope of the graph at x = 1 is greater than 1.

Recall that we earlier concluded that the slope of the graph at x = 1 is less than 2.5. Putting all of our calculations together, what we've learned so far is that the actual slope of the graph at x = 1 is greater than 1, but less than 2.5. Now *this* is a useful approximation, because we have a good sense of the accuracy of the approximation. At least we know that the true value of the slope at x = 1 is between the values of 1 and 2.5.

This approximation can be made even better by moving the second point used to calculate the secant line ever closer to P. Let's try this from both the left and right, summarizing the calculations in a table (the first two lines repeat the calculations done earlier for convenience):

$x_1$	$y_1$	$x_2$	$y_2$	$x_2 - x_1$	$y_2 - y_1$	$\frac{y_2 - y_1}{x_2 - x_1}$
1	1	2.0	4.0	1.0	3.0	3.0
1	1	1.5	2.25	0.5	1.25	2.5
1	1	1.1	1.21	0.1	0.21	2.1
1	1	1.01	1.0201	0.01	0.0201	2.01

Table 1: Overestimates for the slope of the graph of  $f(x) = x^2$  at x = 1.

$x_1$	$y_1$	$x_2$	$y_2$	$x_2 - x_1$	$y_2 - y_1$	$\frac{y_2 - y_1}{x_2 - x_1}$
1	1	-0.5	0.25	-1.5	-0.75	0.5
1	1	0.0	0.0	-1.0	-1.0	1.0
1	1	0.5	0.25	-0.5	-0.75	1.5
1	1	0.9	0.81	-0.1	-0.19	1.9
1	1	0.99	0.9801	-0.01	-0.0199	1.99

Table 2: Underestimates for the slope of the graph of  $f(x) = x^2$  at x = 1.

What we can conclude from the two tables is that the actual slope of the graph of  $f(x) = x^2$  at the point P(1,1) is between 1.99 and 2.01. We still don't know exactly what the slope is, but at least we've pinned it down to between two numbers that are relatively close together.

Even better, we now have a systematic procedure for making the approximation better and better, so that if we need greater accuracy we can just extend the tables a few more rows.

When you study calculus you will learn how to translate this approximation procedure into the language of algebra, so that you will be able to make the approximations precise.

#### PRACTICE

(Answers below.)

1. Use the process outlined in Example 1 to estimate the slope of the graph of each function at the given point. That is, create a table of overestimates, a table of underestimates, and then state that the true slope is between two of your calculated values.

(a) 
$$y = x^2$$
 at  $P(2, 4)$ 

(b)  $y = x^2$  at P(3,9)

(c)  $y = \sin x$  at P(0,0) (where x is measured in radians)

(d)  $y = \sin x$  at  $P(\pi, 0)$  (where x is measured in radians)

(e)  $y = \sin x$  at  $P(\pi/2, 1)$  (where x is measured in radians)

Answers: 1. Your results should be approximately: (a) 4 (b) 6 (c) 1 (d) -1 (e) 0

The line through point P that has the same slope as the true slope of the graph of y = f(x)at P is called the tangent line to the graph of f at the point P. The secant lines approach the tangent line more and more closely as the process in Example 1 is carried out. The tangent line to the graph of  $f(x) = x^2$  at the point P(1, 1) is sketched in Figure 4.



Figure 4: The tangent line to the graph of  $f(x) = x^2$  at the point P(1,1) is the best linear approximation to the graph near P; that is, it is the line through P that has the same slope as the actual slope of the graph at P.

#### **RECAP OF FOCUS QUESTION**

Recall the focus question, which was asked earlier in the lesson.

A bacterial culture grows so that the number N(t) of cells at time t hours after an experiment begins is  $N(t) = 3t^2 + 100$ . Estimate the rate at which the bacterial colony grows 4 hours after the experiment begins.

#### SOLUTION

The rate at which the bacterial colony grows after 4 hours is equal to the slope of the graph of the function N(t) at t = 4. Note that  $N(4) = 3(4^2) + 100 = 148$ . Therefore, we need to calculate the slope of the graph at the point P(4, 148).

Following the procedure outlined in this module, let's calculate the slopes of secant lines PQ, where Q is a point on the graph near to P. We can organize our calculations in two tables, one for the overestimates, one for the underestimates. Which points shall we choose? There is no guidance here, so we'll just choose points on the graph on each side of P that gradually get closer and closer to P. There's also no guidance about how far to take the tables, so we'll just go for a few rows and draw an approximate conclusion.<sup>*a*</sup>

Here are the results; the first table contains the overestimates and the second table contains the underestimates:  $^{b}$ 

Overestimates:

$t_1$	$N_1$	$t_2$	$N_2$	$t_2 - t_1$	$N_2 - N_1$	$\frac{N_2 - N_1}{t_2 - t_1}$
4	148	5.0	175.0	1.0	27.0	27.0
4	148	4.5	160.75	0.5	12.75	25.5
4	148	4.1	150.43	0.1	2.43	24.3
4	148	4.01	148.2403	0.01	0.2403	24.03

Underestimates:

$t_1$	$N_1$	$t_2$	$N_2$	$t_2 - t_1$	$N_2 - N_1$	$\frac{N_2 - N_1}{t_2 - t_1}$
4	148	3.0	127	-1.0	-21.0	21.0
4	148	3.5	136.75	-0.5	-11.25	22.5
4	148	3.9	145.63	-0.1	-2.37	23.7
4	148	3.99	147.7603	-0.01	-0.2397	23.97

Based on the calculations in the table, we can't be sure what the true slope of the graph is at P(4, 148). However, we can be sure that the slope is between 23.97 and 24.03. Therefore, to within about a tenth of a percent uncertainty, we can say that the slope is approximately 24. This means that the bacterial culture is growing at a rate of approximately 24 cells per hour at the 4-hour mark in the experiment.

<sup>&</sup>lt;sup>a</sup>Once you study calculus you'll learn how to take these calculations to the limit to get precise conclusions instead of just approximations.

<sup>&</sup>lt;sup>b</sup>How can we be sure that all of the slope values in the first table are overestimates, and all of the slope values in the second table are underestimates? That is a good question. It has something to do with the fact that the graph of N(t) is concave up. Sketch the graph and see if you can convince yourself of this.

#### EXERCISES

- 1. Approximate the instantaneous rate of change of each function at the given point.
  - (a)  $f(x) = -2x^2 + 4$  at P(1,2)(b)  $f(x) = -x^2 + 3x$  at P(2,2)(c)  $f(x) = \frac{1}{x}$  at  $P(2,\frac{1}{2})$ (d)  $f(x) = 2^x$  at P(1,2)
  - (e)  $f(x) = x^3$  at P(-1, -1)
- 2. For the function whose graph is in the figure, draw secant lines over each interval: [1,3], [1,2.5], [1,2] and [1,1.5]. At which point do the slopes of these secant lines approximate the instantaneous rate of change of the function? Is the instananeous rate of change at this point positive or negative?



- 3. Does each sentence describe an average rate of change or an instantaneous rate of change?
  - (a) If I ran 10 km in 5 minutes, my speed was 2 km/min.
  - (b) At 2 pm, the bacterial colony's population was doubling in size.
  - (c) The slope of the blue line at the blue dot is -1.

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(d) The slope of the line joining the points (1, -3) and (4, 3) on the graph of a function is 2.

## WWW

- What we did: In this module we used graphs and tables of values to investigate the relationship between instantaneous rates of change and average rates of change. We explored a systematic procedure for using increasingly accurate average rates of change to approximate instantaneous rates of change.
- Why we did it: Instantaneous rates of change are a very important concept in calculus and they provide us with a more precise measurement than using an average rate of change.
- What's next: In a subsequent module you will learn how to translate the numerical procedure described in this module into algebraic form, and to calculate precise instantaneous rates of change using the concept of a limit.