

# BROCK UNIVERSITY MATHEMATICS MODULES

## 11A2.4: Maximum or Minimum Values for Quadratic Functions

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### WWW

- What it is: Maximum or minimum values for a quadratic function are the largest and smallest  $y$ -values on its graph.
- Why you need it: We use the maximum and minimum values to analyse the graphs of quadratic functions (which are parabolas) and understand their properties.
- When to use it: Many real-life situations can be modeled by quadratic functions; calculating maximum or minimum values can help us to better understand these situations.

### PREREQUISITES

Before you tackle this module, make sure you have completed these modules:

**Basic knowledge of quadratic functions, Determining Zeros of a Quadratic Function (11A2.1), Completing the Square (11A2.3), The Quadratic Formula**

### WARMUP

Before you tackle this module, make sure you can solve the following exercises. If you have difficulties, please review the appropriate prerequisite modules.

(Answers below.)

1. State whether each parabola opens upwards or opens downwards.

(a)  $y = -5x^2 + 6x - 9$    (b)  $y = 2x^2 - 4x + 1$    (c)  $y = 3x^2 - 6x + 6$    (d)  $y = -8x^2 + 4x - 5$

2. Complete the square to convert each formula from standard form to vertex form.

(a)  $y = x^2 + 8x$    (b)  $y = x^2 - 2x - 5$    (c)  $y = 3x^2 + 4x - 2$

3. Graph each function using a table of values.

(a)  $y = 2x^2$    (b)  $y = -2x^2$

4. Calculate the zeros for each function.

(a)  $y = x^2 - x - 6$    (b)  $y = 6x^2 + x - 2$

5. Use the quadratic formula to calculate the zeros of each function.

(a)  $y = x^2 - 3x - 2$     (b)  $y = 4x^2 - 7x + 1$

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Answers: 1.(a) opens downwards (b) opens upwards (c) opens upwards (d) opens downwards 2.(a)  $(x+4)^2 - 16$  (b)  $(x-1)^2 - 6$  (c)  $3(x+2/3)^2 - 10/3$  3. The graphs resemble the ones given in Figures 1 and 2 on the following page. 4.(a)  $-2, 3$  (b)  $-2/3, 1/2$  5.(a)  $\frac{3 \pm \sqrt{17}}{2}$  (b)  $\frac{7 \pm \sqrt{33}}{8}$

## Introduction

Functions are used extensively to model various aspects of our world, and so are important for science, engineering, business, economics, and other fields. One of the major reasons for studying functions in high school (and university) is to prepare us to use functions for modeling. To this end, it is valuable to have a good understanding of some basic functions. The simplest functions are linear; the next simplest are quadratic. Having mastered these, one then continues with more complex classes of functions, such as polynomial, rational, trigonometric, exponential, logarithmic, and so on. Once one reaches a certain level, calculus becomes an extremely helpful tool for analyzing functions. By mastering a few basic types of functions, you will prepare yourself to take the next step towards learning calculus.

The maximum and minimum values of a function are often the most interesting values in an applied problem. In this module, we focus on learning how to calculate the maximum or minimum value for quadratic functions, which have parabolas as graphs. For example, the paths of projectiles close to the earth's surface are approximately parabolas, and the position-time graphs of objects projected vertically upwards are also approximately parabolas. Besides motion problems, many other situations in science, business, and economics can be modeled using quadratic functions.

The position-time graph of a projected object near the earth's surface is approximately parabolic (not exactly so, because air resistance is always present, and because of a slight decrease in the strength of gravity as you move away from the earth).

### FOCUS QUESTION

To help you understand an important aspect of this lesson, focus your attention on this question, which will be answered towards the end of the lesson.

**A baseball is launched upwards at an initial speed of 19.6 m/s from a platform that is 58.8 m high. The equation for the object's height,  $s$ , at time,  $t$  seconds, after launch is  $s(t) = -4.9t^2 + 19.6t + 58.8$ , where  $s$  is in metres. What is the greatest height the ball reaches?**

Recall that the graph of a quadratic function that has formula  $y = ax^2 + bx + c$  is a parabola that either opens upward or opens downward; in other words the axis of symmetry is a vertical line. The parabola opens upward if  $a > 0$ , and it opens downward if  $a < 0$ . See Figures 1 and 2 for the simplest examples.

Notice that for the parabola in Figure 1, which opens up, there is a minimum value (which is the  $y$ -value at the vertex, namely 0), but no maximum value. For the parabola in Figure 2, which opens down, there is a maximum value (which is the  $y$ -value at the vertex, also 0), but no minimum value.

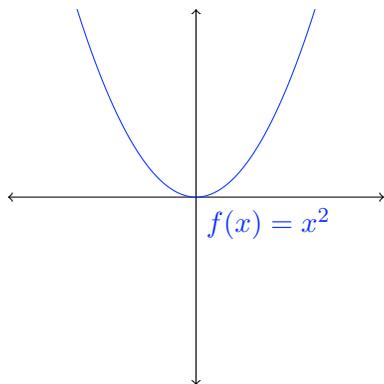


Figure 1: Graph of a parabola opening up.

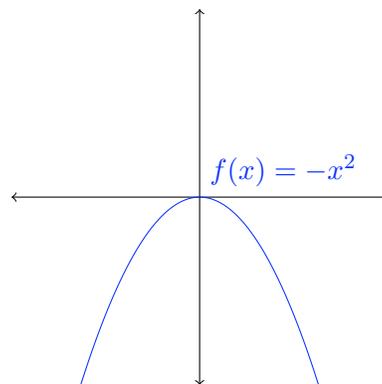


Figure 2: Graph of a parabola opening down.

The examples in the figures are the simplest possible. But how can we calculate the maximum or minimum value of a parabola in general, when the coordinates of the vertex are not so simple?

There are a few different ways to calculate the maximum or minimum value of a parabola. Since we are after the  $y$ -coordinate of the vertex, one way is to place the formula into vertex form; then you can read the coordinates from the formula.

If the parabola has two  $x$ -intercepts, another way to determine the coordinates of the vertex is to first determine the values of the  $x$ -intercepts by setting  $y = 0$  and solving for  $x$ . Then the  $x$ -coordinate of the vertex is the average of the two  $x$ -intercepts. (If a parabola is reflected about its axis of symmetry, it is the same; this means that if there are two  $x$ -intercepts they will be the same distance from the axis of symmetry.) From this you can calculate the  $y$ -coordinate of the vertex by substituting the  $x$ -coordinate of the vertex into the formula for  $y$ .

Yet another way to determine the coordinates of the vertex is to use calculus, but we'll save that for another time! Instead, let's illustrate the first two methods for calculating the maximum or minimum values of a parabola with some examples.

### EXAMPLE 1

Determine the maximum or minimum value of the quadratic function  $y = 4x^2 + 8x - 3$ .

### SOLUTION

Using the strategy of placing the quadratic function's formula in vertex form, here are the steps we need to perform:

1. Place the formula in vertex form.
2. Determine the coordinates of the vertex.
3. Note whether the parabola opens up or down.
4. State a conclusion.

So to start things off, the way you get this equation into vertex form is by completing the square.<sup>a</sup> At the end of this process, the formula will be in the form  $y = a(x - h) + k$ , and the vertex is  $(h, k)$ . Let's do this!

$$\begin{aligned}y &= 4x^2 + 8x - 3 \\y &= 4(x^2 + 2x) - 3 \\ \frac{2}{2} &= 1 \quad \text{and} \quad 1^2 = 1 \\y &= 4(x^2 + 2x + 1 - 1) - 3 \\y &= 4(x^2 + 2x + 1) - 4 - 3 \\y &= 4(x^2 + 2x + 1) - 7 \\y &= 4(x + 1)^2 - 7\end{aligned}$$

This is the vertex form of the formula; as you can see, the coordinates of the vertex are  $h = -1$  and  $k = -7$ . This means that the vertex is at  $(-1, -7)$ . Remember, the  $y$ -value of this vertex is either our maximum or minimum value.

Next, notice that the coefficient of  $x^2$  in the function's formula is positive (4 to be exact). This means that the parabola opens up, so the  $y$ -coordinate of the vertex represents a minimum value.

Therefore, the minimum value of the quadratic function  $y = 4x^2 + 8x - 3$  is  $y = -7$ . You can confirm this by plotting a graph of the function.

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<sup>a</sup>See Module 11A2.3 for a review of completing the square.

## EXAMPLE 2

Determine the maximum or minimum value of the quadratic function  $y = -2x^2 - 8x + 9$ .

### SOLUTION

Let's complete the square, as we did in the previous example:

$$\begin{aligned}y &= -2x^2 - 8x + 9 \\y &= -2(x^2 + 4x) + 9 \\ \frac{4}{2} &= 2 \quad \text{and} \quad 2^2 = 4 \\y &= -2(x^2 + 4x + 4 - 4) + 9 \\y &= -2(x^2 + 4x + 4) + 8 + 9 \\y &= -2(x^2 + 4x + 4) + 17 \\y &= -2(x + 2)^2 + 17\end{aligned}$$

This means that the vertex is at  $(-2, 17)$ . Since the coefficient of  $x^2$  is negative, the parabola opens downward. Thus, the parabola has a maximum value of  $y = 17$ .

To confirm this conclusion, plot a graph of the function.

Now try these exercises on your own!

## PRACTICE

(Answers below.)

1. Determine the maximum or minimum value for each quadratic function.

(a)  $y = 3x^2 + 6x - 2$       (b)  $y = -8x^2 - 24x + 9$

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Answers: 1.(a) minimum value of  $-5$ ; (b) maximum value of  $27$

The following example shows how to calculate the coordinates of the vertex of a parabola using the  $x$ -intercepts.

### EXAMPLE 3

Determine the maximum or minimum value of the quadratic function  $y = x^2 - 2x - 15$  using its  $x$ -intercepts.

### SOLUTION

Here's the strategy we'll follow to solve this problem:

1. Determine the  $x$ -intercepts.
2. Determine the  $x$ -value of the vertex.
3. Determine the  $y$ -value of the vertex.
4. Notice whether the parabola opens up or down.
5. State a conclusion.

To determine the  $x$ -intercepts, set  $y = 0$  and solve for  $x$ .<sup>a</sup>

$$\begin{aligned}0 &= x^2 - 2x - 15 \\0 &= (x + 3)(x - 5)\end{aligned}$$

Thus, there are two  $x$ -intercepts,  $x_1 = -3$  and  $x_2 = 5$ . The  $x$ -intercepts are equally spaced on either side of the axis of symmetry, because of the symmetry properties of parabolas. Therefore, the  $x$ -coordinate of the vertex is the average of the  $x$ -intercepts:

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<sup>a</sup>For a review of calculating zeros, see Module 11A2.1.

$$\begin{aligned}
 x\text{-coördinate of vertex} &= \frac{x_1 + x_2}{2} \\
 &= \frac{-3 + 5}{2} \\
 &= \frac{2}{2} \\
 &= 1
 \end{aligned}$$

Therefore, the  $x$ -coördinate of the vertex is  $x = 1$ . (Sketch a number line to verify that the  $x$ -coördinate of the vertex is the same distance from each  $x$ -intercept.) To determine the  $y$ -coördinate of the vertex, just substitute  $x = 1$  into the formula for the parabola:

$$\begin{aligned}
 y &= x^2 - 2x - 15 \\
 y &= (1)^2 - 2(1) - 15 \\
 y &= -16
 \end{aligned}$$

Therefore, the  $y$ -coördinate of the vertex is  $-16$ . In other words, the vertex is located at the point  $(1, -16)$ .

Finally, notice that the coefficient of  $x^2$  in the formula for the parabola is positive. This means the parabola opens up, and so it has a minimum value. Therefore, the minimum value of the quadratic function  $y = x^2 - 2x - 15$  is  $-16$ .

What if there are no  $x$ -intercepts? You could adapt the idea behind the previous example, using some other horizontal line instead of the  $x$ -axis. Explore this if you're interested! But it might be easier to complete the square, as in the first two examples.

The following exercises will give you some practice in using  $x$ -intercepts to determine the maximum or minimum value of a parabola:

### PRACTICE

(Answers below.)

2. Determine the maximum or minimum value of each parabola using the  $x$ -intercepts.

(a)  $y = x^2 - 6x + 5$       (b)  $y = -2x^2 + 12x - 16$

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Answers: 2.(a) minimum of  $-4$ ;    (b) maximum of  $2$

Let's now use what you've learned to an applied example.

#### EXAMPLE 4

Part of a roller coaster track is parabolic, and can be modelled by the quadratic function  $y = 0.4x^2 - 0.6x + 0.8$ , where  $x$  represents the horizontal distance from a marker,  $y$  represents the height above the ground, and  $x$  and  $y$  are both measured in metres.

Determine the minimum height above the ground of this part of the roller coaster track.

#### SOLUTION

The coefficient of  $x^2$  is positive, so this parabolic section of track really does have a minimum height. Let's complete the square:

$$\begin{aligned}y &= 0.4x^2 - 0.6x + 0.8 \\&= 0.4(x^2 - 1.5x) + 0.8 \\ \text{Note that } \frac{-1.5}{2} &= -0.75, \text{ and } (-0.75)^2 = 0.5625. \text{ Therefore,} \\y &= 0.4(x^2 - 1.5x + 0.5625 - 0.5625) + 0.8 \\&= 0.4(x^2 - 1.5x + 0.5625 - 0.5625) + 0.8 - (0.4)(0.5625) \\&= 0.4(x^2 - 1.5x + 0.5625) + 0.8 - 0.225 \\&= 0.4(x - 0.75)^2 + 0.575\end{aligned}$$

From the previous line, we can read off the minimum value: The height of the parabolic section of track reaches a minimum height of 0.575 m above the ground, and this minimum height occurs at a distance of 0.75 m from the marker.

Was it difficult to follow the previous calculation because of the decimal numbers? Let's repeat the calculation using fractions; then you can compare the two calculations and decide for yourself which you prefer:

$$\begin{aligned}y &= 0.4x^2 - 0.6x + 0.8 \\&= \frac{2}{5}\left(x^2 - \frac{3}{2}x\right) + \frac{4}{5} \\ \text{Note that } \frac{-3}{2} &= \frac{-3}{4}, \text{ and } \left(\frac{-3}{4}\right)^2 = \frac{9}{16}. \text{ Therefore,} \\y &= \frac{2}{5}\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right) + \frac{4}{5} \\&= \frac{2}{5}\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + \frac{4}{5} - \frac{2}{5} \cdot \frac{9}{16} \\&= \frac{2}{5}\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + \frac{4}{5} - \frac{9}{40} \\&= \frac{2}{5}\left(x - \frac{3}{4}\right)^2 + \frac{32}{40} - \frac{9}{40} \\&= \frac{2}{5}\left(x - \frac{3}{4}\right)^2 + \frac{23}{40}\end{aligned}$$

If you convert the fractions in the previous line to decimals, you will see that the result of this calculation is exactly the same as the result of the previous one. Which one do you find easier?

Now let's finish up by answering the focus question, if you haven't already done so.

### RECAP OF FOCUS QUESTION

Recall the focus question, which was asked earlier in the lesson.

A baseball is launched upwards at an initial speed of 19.6 m/s from a platform that is 58.8 m high. The equation for the object's height,  $s$ , at time,  $t$  seconds, after launch is  $s(t) = -4.9t^2 + 19.6t + 58.8$ , where  $s$  is in metres. What is the greatest height the ball reaches?

### SOLUTION

First complete the square:

$$\begin{aligned} s(t) &= -4.9t^2 + 19.6t + 58.8 \\ &= -4.9(t^2 - 4t) + 58.8 \\ &= -4.9(t^2 - 4t + 4 - 4) + 58.8 \\ &= -4.9(t^2 - 4t + 4) + 4.9(4) + 58.8 \\ &= -4.9(t^2 - 4t + 4) + 78.4 \\ &= -4.9(t - 2)^2 + 78.4 \end{aligned}$$

The vertex is at (2, 78.4). Because the coefficient of  $t^2$  is negative, the quadratic function has a maximum value. Thus, the maximum height of the ball is 78.4 m above the ground. This height is attained 2 s after the ball is thrown.

### EXERCISES

3. Juliet stands on the balcony of her third floor apartment. She leans out and throws a ball straight up with an initial speed of 20 m/s.<sup>a</sup> The ball's height  $y$  in metres above the ground  $t$  seconds after it is thrown can be approximated by the formula  $y = -5t^2 + 20t + 11.25$ .

When does the ball reach its maximum height above the ground? What is the maximum height?

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<sup>a</sup>Yes, as you expected, Romeo is standing on the ground, waiting to catch the ball.

### WWW

- What we did: We learned how to determine the maximum or minimum value of a quadratic function.
- Why we did it: We did this to better understand many real life situations that can be modeled by quadratic functions.
- What's next: When we learn calculus, we can solve similar problems for more complex functions.