BROCK UNIVERSITY MATHEMATICS MODULES

11A1.5: Inverse of a Function

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WWW

- What it is: A function f takes an input x and produces an output y. The function that is the inverse of f reverses this process, by taking y as its input and producing the output x.
- Why you need it: The idea of an inverse function helps us to solve certain problems, and it also helps to give us another perspective on a relation.
- When to use it: Use an inverse function when you know the output value of a function and you need to solve for the input value. For example, if you know that $\sin(\theta) = 0.5$, the inverse sine function will help you determine the corresponding value of θ .

PREREQUISITES

Before you tackle this module, make sure you have completed these modules:

Vertical line test, Function notation, Graphs of linear and quadratic functions, Solving linear and quadratic equations

WARMUP

Before you tackle this module, make sure you can solve the following exercises. If you have difficulties, please review the appropriate prerequisite modules.

(Answers below.)

1. For the function $f(x) = 3x^2 - 2x + 5$, determine each quantity.

(a)
$$f(0)$$
 (b) $f(-1)$ (c) $f(1/2)$

2. Solve each equation for x.

(a) 3x + 1 = 10 (b) -3x + 4 = -2 (c) -2x - 5 = 1

3. Determine real values of x that are solutions for each equation.

(a) $x^2 - 3x + 2 = 0$ (b) $x^2 - 3x + 1 = 0$ (c) $x^2 - 3x + 3 = 0$



Introduction

There are a number of perspectives on functions, each useful in certain situations. For example, we might think of a function in terms of its formula (if it has one), or its graph, or as a verbal description of a relationship between the variables.

Another way to think of a function is *operationally*; we think of a function as a kind of "black box," where we input a number. The function inside the box then does some operations on the the input number and delivers a number as the output from the box. This is the same thing that happens when you do a calculation using your calculator; you input a number, do a bunch of operations, and then there is a final number on your calculator screen, which is the output.

This "black box" perspective of a function is illustrated schematically in the following figure:

(input)
$$x \longrightarrow f \longrightarrow y$$
 (output)

Figure 1: An operational perspective of the function y = f(x).

For example, consider the function f(x) = 3x + 2, which we could also write as y = 3x + 2. The name of the function is f, input values are represented by x, and output values are represented by y. Verbally, the action of the function f can be described as follows: an input value is multiplied by 3, then 2 is added to the result to obtain the output value. For example, if 1 is the input value (that is, x = 1), then the output value is 3(1) + 2 = 5 (that is, y = 5). We can represent this using a "black box" diagram as follows:



Figure 2: For the function f(x) = 3x + 2, when the input is x = 1, the output is y = 5.

By inputting many values into the f box, and studying the corresponding outputs, we would come to some understanding of the function f.

The idea behind inverse functions is to run values through the f box backwards. We would like to understand how the black box behaves when we run it backwards. For example, see Figure 3.

(output)
$$1 \leftarrow f^{-1} \leftarrow 5$$
 (input)

Figure 3: For the function f(x) = 3x + 2, the inverse function is denoted f^{-1} . When the input to the inverse function box is 5, the output is 1, reversing the action of the function f.

We would have a concrete understanding if we could obtain a formula for the inverse operation. One of the goals of this module is to learn how to obtain a formula for the inverse function starting from a formula for the function. But before we do that, we can already see that the operations of the box change when we run it backwards. The operations must change in a very specific way so that we get back to our starting point when we run the box backwards. This is why we label the box differently $(f^{-1} \text{ instead of } f)$ when we run the box backwards.

Specifically, the actions of the function f and the inverse function f^{-1} in Figures 2 and 3 can be expressed symolically as follows: If f(1) = 5, then $f^{-1}(5) = 1$. The f function takes the input 1 and produces the output 5, and then the f^{-1} function reverses the process, taking the input 5 and returning us to our starting place by producing the number 1 as an output. One of the uses of inverse functions is to help us determine values of a function that produce specific outputs. This helps us to solve equations involving functions, and is used frequently in mathematics and its applications.

The idea of an inverse function also helps us to understand a functonal relationship from a reverse perspective. For example, suppose you are flying a kite. You might determine a relationship between the height of the kite and how windy it is. Careful measurments might allow you to construct a formula that relates the kite's height to the wind speed, so that if you know the wind speed you can predict the height of the kite. The inverse functional relationship would allow you to measure the kite's height and then use it to determine the wind speed.

In this module you'll see examples of functions and their inverses, from various perspectives, and learn how to determine the inverse of a function numerically, graphically, and algebraically.

FOCUS QUESTION

To help you understand an important aspect of this lesson, focus your attention on this question, which will be answered towards the end of the lesson.

The I.T. department at Superfluous Superfluids Inc. installs computer equipment. The amount charged for installing x computers in a year is (in thousands of dollars) f(x) = 2x + 50. An accountant is brought in to audit the department's records, which claim that they spent \$870 000 last year for computer installations.

(a) How many computers were installed?

(b) If the auditor had to perform many calculations such as the one in Part (a), which formula could he use?

Let's suppose you have an interview for a summer job. This job pays \$10 per hour for as many hours as you wish to work, and the company pays you at the end of each day. To calculate the amount of money you would earn in a day, you could use the function f(x) = 10x, where x is the number of hours you work and f(x) is the amount of money (in dollars) you earn. For example, if you work 6 hours one day, you will earn f(6) = (10)(6) = \$60. You can see the results for some numbers of hours worked in the following table of values for the function.

| x | f(x) |
|---|------|
| 0 | 0 |
| 1 | 10 |
| 2 | 20 |
| 3 | 30 |
| 4 | 40 |
| 5 | 50 |
| 6 | 60 |
| 7 | 70 |
| 8 | 80 |

Suppose that your goal is to make a certain amount of money in a day; how many hours should you work to achieve your goal? The answer can be worked out by obtaining a formula for the inverse function. But before we do this, let's think about the operational perspective of a function. The inverse of a function just means to run the function's "black box" backwards. This means to treat the outputs of f as the inputs of f^{-1} and vice versa. This means that if we reverse the table of values for the function, we'll get a table of values for the inverse function:

| x | $f^{-1}(x)$ |
|----|-------------|
| 0 | 0 |
| 10 | 1 |
| 20 | 2 |
| 30 | 3 |
| 40 | 4 |
| 50 | 5 |
| 60 | 6 |
| 70 | 7 |
| 80 | 8 |

If you wish to make \$70 for a day's work, you can read from the table the number of hours you would have to work to achieve that goal: 7 hours.

As you can see, the inverse function contains the same information as the original function, but gives you a different perspective.

Now let's think of the inverse function in terms of operations. If you start with a number, apply the operations of a function, and then continue to apply the operations of the inverse function, you will end up back at your starting point. Let's look at an example to see this. Suppose you begin with the number 4 as your input. The operations of the function g is to take the input, multiply it by 3, and then add 7. Thus, g(4) = (3)(4) + 7 = 12 + 7 = 19. In a table of values for the function g, one row of the table would be

$$\begin{array}{c|c} x & g(x) \\ \hline 4 & 19 \end{array}$$

Now let's look at g's inverse function in terms of operations. To determine the action of g^{-1} we have to reverse each of the operations of the function g. We must also be careful to reverse them in the *opposite order*.¹ For example, take your bare feet, then put socks on, and finally put shoes on. You can think of this sequence of operations as a function, which takes bare feet as input and produces feet with socks and shoes on. The inverse function takes feet with socks and shoes on and produces bare feet. The inverse function first takes the shoes off (reversing the "putting shoes on" operation) and follows this by taking the socks off (reversing the "putting socks on" operation). Notice that the operations were reversed, and their order was also reversed.

The same process applies to any function when you determine its inverse, no matter how many operations are strung together. You reverse each operation, but in opposite order. For the g function in the previous two paragraphs, the inverse operation is to subtract 7, then divide by 3:

$$g \longrightarrow$$
 multiply by 3 and then add 7
 $g^{-1} \longrightarrow$ subtract 7 and then divide by 3

Note that:

$$g(4) = (3)(4) + 7 = 19$$

 $g^{-1}(19) = (19 - 7)/3 = 4$

This verifies that the second function really is the inverse of the first one. Applying g to 4 produces 19, and then applying g^{-1} to 19 brings us back to the starting number, which is 4.

¹Does this mean that if we use the BEDMAS convention for order of operations for the function, must we use SAMDEB for the inverse function? It has a nice ring to it, I think ... SAMDEB ... very nice!

Study the previous equations to conclude that we can write the actions of g and g^{-1} as formulas as follows:

$$g(x) = 3x + 7$$
$$g^{-1}(x) = \frac{x - 7}{3}$$

The fact that applying g to an input x, followed by applying g^{-1} brings us back to the starting input value can be expressed as follows:

$$g^{-1}\left(g(x)\right) = x$$

The fact that the previous statement is true for all possible input x-values can be seen as follows, for the example of the function g:

$$g^{-1}(g(x)) = g^{-1}(3x+7)$$

= $\frac{(3x+7)-7}{3}$
= $\frac{3x+7-7}{3}$
= $\frac{3x}{3}$
= x

Our discussion so far about functions and their inverses can be summarized by the following definition:

DEFINITION

The function f^{-1} is the *inverse* of the function f provided that

 $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$

are valid for all x-values in the domain of f and for all y-values in the domain of f^{-1} .

EXAMPLE 1

Construct a table of values for the function y = 2x + 1 and its inverse function.

SOLUTION

Selecting a few x-values, a table of values for the function is

| x | y |
|----|----|
| -2 | -3 |
| -1 | -1 |
| 0 | 1 |
| 1 | 3 |
| 2 | 5 |

Reversing the inputs and outputs of the previous table of values (that is, interchanging the x-column and the y-column) produces a table of values for the inverse function:

| x | y |
|----|----|
| -3 | -2 |
| -1 | -1 |
| 1 | 0 |
| 3 | 1 |
| 5 | 2 |
| | 1 |

Let's try out a few questions to see if we understand what the inverse of a function is!

PRACTICE

(Answers below.)

1. (a) For each function, complete the table of values. Then construct a table of values for the inverse function.

| f(x) = 3x + 1 Inverse function | | g(x) = -5x + 3 | | Inverse function | | | |
|-----------------------------------|---|----------------|---|------------------|---|---|---|
| x | y | x | y | x | y | x | y |
| -2 | | | | -2 | | | |
| -1 | | | | -1 | | | |
| 0 | | | | 0 | | | |
| 1 | | | | 1 | | | |
| 2 | | | | 2 | | | |

(b) Using the tables of values in Part (a), construct the graph of each function and its inverse on the same axes. Include the line y = x on each graph.

Answers: 1. (a)

| f(x) = | 3x + 1 | $f^{-1}(x) =$ | $=\frac{x-1}{3}$ | | g(x) = -5x + 3 | | $g^{-1}(x) = \frac{3-x}{5}$ | |
|--------|--------|---------------|------------------|---|----------------|----|-----------------------------|----|
| x | y | x | y | Π | x | y | x | y |
| -2 | -5 | -5 | -2 | | -2 | 13 | 13 | -2 |
| -1 | -2 | -2 | -1 | | -1 | 8 | 8 | -1 |
| 0 | 1 | 1 | 0 | Π | 0 | 3 | 3 | 0 |
| 1 | 4 | 4 | 1 | Π | 1 | -2 | -2 | 1 |
| 2 | 7 | 7 | 2 | | 2 | -7 | -7 | 2 |

(b) Original function is red, inverse is green





Consider the procedure used in the previous example to produce a table of values for the inverse function by reversing the columns in the table of values for the function. The same procedure can be applied to the formula for a function to produce a formula for the inverse function. That is, we can interchange x and y in the formula for a function, then solve for y to obtain a formula for the inverse function. Let's see how this process works for the function g(x) = 3x + 7 that we studied earlier in this module.

EXAMPLE 2

Determine a formula for $g^{-1}(x)$, the function inverse to g(x) = 3x + 7.

SOLUTION

Following the procedure outlined in the previous paragraph, replace g(x) by y, then interchange x and y, and finally solve for the new y. The formula obtained is the formula for $g^{-1}(x)$.

$$y = 3x + 7$$

$$x = 3y + 7 \qquad \text{(interchange } x \text{ and } y\text{)}$$

$$x - 7 = 3y$$

$$3y = x - 7$$

$$y = \frac{x - 7}{3}$$

This new formula for y is a formula for $g^{-1}(x)$. That is,

$$g^{-1}(x) = \frac{x-7}{3}$$

This formula is exactly the same as the one we determined earlier in this module.

Now that we've worked out a procedure for determining a formula for the inverse of a function, it's worth stating it officially for reference:

KEY IDEA

To determine a formula for y = f(x), interchange x and y, then solve for the new y. The result is a formula for $y = f^{-1}(x)$.

PRACTICE

(Answers below.)

Determine a formula for the inverse function in each case.

- 2. y = 2x + 1
- 3. y = -3x + 4
- 4. y = 5x + 2

5. Determine a formula for the inverse of each function. (a) f(x) = 3x + 4 (b) $f(x) = \sqrt{2x + 5}$ Answers: 2. y = (x - 1)/2 3. y = (4 - x)/3 4. y = (x - 2)/5 5.(a) $f^{-1}(x) = (x - 4)/3$ (b) $f^{-1}(x) = (x^2 - 5)/2$

Now that we've studied the concept of an inverse function operationally (the "black box"), numerically (using tables of values), and algebraically (manipulating the formula for the function to get a formula for the inverse function), let's consider a graphical perspective. Remember that the idea of an inverse function is to reverse a series of operations, which amounts to exchanging the x-column and the y-column of a table of values, which is the same as interchanging x and y in a formula. What does this look like graphically?

A good way to understand the graphical perspective is to sketch the graph of a function and its inverse on the same set of axes. Consider the function f(x) = 2x + 1, which you would have studied in the previous set of practice questions. The following figure shows the graph of f.



Figure 4: Graph of the function f(x) = 2x + 1.

Plotting the graph of the inverse function $f^{-1}(x) = (x-1)/2$, we get the following graph:



Figure 5: Graph of the function $f^{-1}(x) = \frac{x-1}{2}$

Notice anything special about these lines? How are they related? It might help if we plotted them on the same axes; check out the following graph:



Figure 6: Graphs of f and f^{-1} on the same axes. How are the two lines related?

Still no? What if we include the line y = x:



Figure 7: Graphs of f and f^{-1} on the same axes, along with the line y = x. Notice how the graphs of f and f^{-1} are related to each other with respect to the line y = x.

Can you see that the graph of f^{-1} is actually a reflection of the graph of f in the line y = x?

It might be easier to see this if you take a sheet of transparent plastic, such as an acetate sheet that is used on overhead projectors. Draw in the x-axis, the y-axis, and the line y = x. Then, in a different colour, sketch the graph of f. Now, we have learned that the procedure for producing the graph of the inverse function is to interchange x and y. For points along the line y = x, interchanging x and y leaves the points unchanged. This means that the line y = x will not change position in the interchange process.

Now, place each of your hands at opposite ends of the y = x line. Give the acetate sheet a flip so that your hands remain in the same positions. (That is, flip the sheet over.) The line y = x should end up in the same position. Notice that the positive x-axis and the positive y-axis will trade places; this makes sense, because in the process of determining an inverse, x and y are interchanged. The graph of f will move to the position of the graph of f^{-1} . The labels you drew will be on the opposite side of the acetate now, as you have flipped the sheet.

You could also use a plain white piece of paper with a dark marker, and you will see the same result, although it might be a little harder to see through the paper than through a transparent plastic sheet. In any case, we now have a graphical perspective on inverse functions, which is summarized in the following box:

KEY IDEA

The graph of the inverse of a function is a reflection of the graph of the original function in the line y = x.

Let's determine the inverse of a more complex function.

EXAMPLE 3

Graph the following function and its inverse on the same axes.

$$f(x) = \frac{1}{x+1}$$

SOLUTION

Let's first look at the graph and see if we can just easily flip it across the line y = x.



To help us see what the graph of the inverse looks like, it's helpful to identify some key features of the graph of f, and to note what happens to them when the graph of f is flipped in the line y = x. The dotted line represents a vertical asymptote, which is reasonable because $x \neq -1$. We can also see that $y \neq 0$, because the numerator in the formula for y cannot be 0. This means we also have a horizontal asymptote along the x-axis, which also seems apparent from the graph.

What happens to the asymptotes when the graph of f is reflected in the line y = x? Remember that x and y are interchanged in determining the inverse; this means that:

- x = -1 is a vertical asymptote for $f \implies y = -1$ is a horizontal asymptote for f^{-1} y = 0 is a horizontal asymptote for $f \implies x = 0$ is a vertical asymptote for f^{-1}

With these points in mind, it will be easier to sketch the graph of f^{-1} . First let's determine a formula for f^{-1} :

$$f(x) = \frac{1}{x+1}$$

$$y = \frac{1}{x+1}$$

$$x = \frac{1}{y+1} \quad \text{(interchange } x \text{ and } y\text{)}$$

$$y+1 = \frac{1}{x}$$

$$y = \frac{1}{x} - 1$$

$$^{-1}(x) = \frac{1}{x} - 1$$

Note that the formula for f^{-1} confirms our arguments about the asymptotes in the previous paragraph. The graph of f^{-1} also confirms this:

f



Sketching the graphs of both f and f^{-1} on the same set of axes allows us to observe more clearly that they are indeed reflections of one-another in the line y = x:



Now we have several different perspectives on inverse functions, but they are all essentially equivalent. Running a function's "black box" backwards, interchanging two columns in a table of values, interchanging x and y in a formula, and reflecting the graph of a function in the line y = x all amount to the same thing.



RECAP OF FOCUS QUESTION

Recall the focus question, which was asked earlier in the lesson.

The I.T. department at Superfluous Superfluids Inc. installs computer equipment. The amount charged for installing x computers in a year is (in thousands of dollars) f(x) = 2x + 50. An accountant is brought in to audit the department's records, which claim that they spent \$870 000 last year for computer installations.

(a) How many computers were installed?

(b) If the auditor had to perform many calculations such as the one in Part (a), which formula could he use?

SOLUTION

Let's solve Part (b) first, then use the resulting formula to solve Part (a).

To determine a formula for the number of computers installed for a given amount charged, we need the inverse function. As usual, we'll interchange x and y and solve for the new y:

$$f(x) = 2x + 50$$

$$y = 2x + 50$$

$$x = 2y + 50 \quad \text{(interchange } x \text{ and } y\text{)}$$

$$x - 50 = 2y$$

$$2y = x - 50$$

$$y = \frac{x - 50}{2}$$

$$f^{-1}(x) = \frac{x - 50}{2}$$

The previous formula is the solution to Part (b). Remember that in the previous formula the amount charged is in thousands of dollars. Therefore, the number of computers installed can be calculated by using x = 870 in the previous formula:

$$f^{-1}(x) = \frac{x - 50}{2}$$
$$f^{-1}(870) = \frac{870 - 50}{2}$$
$$= \frac{820}{2}$$
$$= 410$$

Thus, the I.T. department claimed to install 410 computers. The auditor can now check to see if this was really the case.

DISCUSSION PROBLEM

The following problem is open in the sense that there may be no definitive solution. Unlike typical textbook exercises, real-life problems rarely have cut-and-dried solutions. Discuss this problem with classmates or friends, then do your best to come up with a reasonable solution, and be prepared to identify and defend the assumptions you make.

Relating the Slopes of the Graphs of a Function and its Inverse

You have seen in this module how the graph of a function and its inverse are related, and also how their formulas are related. If their graphs have a certain relation, it's reasonable to guess that the slopes of the graphs of a function and its inverse are also related.

Explore this possibility. As usual in mathematics, always start with the simplest possible examples. You have already graphed several functions and their inverses in this module; start with the linear functions and their inverses and see if their slopes are related. Do your best to describe any relation you may find as precisely as you can, both in terms of formulas and also in words.

Once you've sorted out the situation for linear functions and their inverses, explore the situation for non-linear functions and their inverses. Have fun!

WWW

- What we did: We learned several perspectives on the inverse of a function: operational, numerical, graphical, and algebraic. We also learned how to use each of the perspectives to determine the inverse of a function.
- Why we did it: Having various perspectives on the inverse of a function allows us to solve problems related to the function using the method that is most appropriate.
- What's next: We can now continue to learn how to determine inverses of more complicated functions, such as higher-degree polynomials, trigonometric functions, and exponential and logarithmic functions.

EXERCISES

- 1. Complete the following inverse function sentence. For example, if a function answers the question, "How much money does Alice earn by working for x hours?" then the inverse function answers the question, "How long does Alice work if she earns y?"
 - (a) If a function answers the question, "How much do x spoons weigh?" which question does the inverse function answer?
 - (b) If a function answers the question, "How many hours of music fit on x CDs?" then which question does the inverse function answer?
- 2. Suppose that there are x boys and y girls in a classroom, and that the number of boys is related to the number of girls by y = 3 + 4x. Determine the inverse function so that one can relate the number of girls to the number of boys.