

BROCK UNIVERSITY MATHEMATICS MODULES

11A1.4: Domain and Range

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WWW

- What it is: The domain refers to the collection of all acceptable input values for a function. The range refers to the collection of all output values for a function, given the acceptable input values.
- Why you need it: In solving problems, candidate solutions sometimes lie outside the domain of a function and therefore must be rejected. Similarly, the domain and range of a function can help us to decide whether a function is a suitable model for the phenomenon being studied.
- When to use it: Use the domain of a function to decide whether a certain value is an acceptable input to a function. Use the range to decide whether a certain value is an acceptable output from a function.

PREREQUISITES

Before you tackle this module, make sure you have completed these modules:

Graphing functions, Factoring quadratic expressions, Quadratic formula, Square root functions, Rational functions, Function notation

WARMUP

Before you tackle this module, make sure you can solve the following exercises. If you have difficulties, please review the appropriate prerequisite modules.

(Answers below.)

1. Factor each quadratic expression.

(a) $x^2 + 5x + 6$ (b) $x^2 - 5x + 6$ (c) $x^2 + 5x - 6$

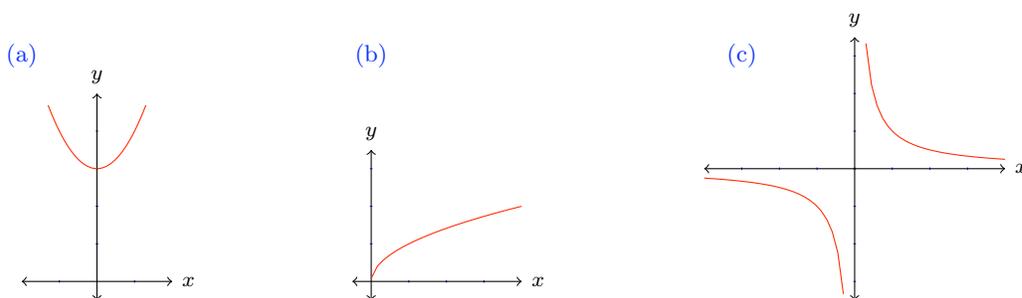
(d) $x^2 + 6x + 5$ (e) $2x^2 + x - 6$ (f) $4x^2 + 5x - 6$

2. Use the quadratic formula to solve each equation.

(a) $x^2 - x - 1 = 0$ (b) $2x^2 + 3x - 4 = 0$ (c) $x^2 + x + 2 = 0$

3. Are the functions $y = 3x - 6$ and $g(x) = 3x - 6$ the same?
4. Consider the function $h(x) = 3x^2 + x - 5$. Determine each value.
- (a) $h(x)$ when $x = -2$ (b) $h(1)$ (c) $h(2)$
5. Sketch a graph of each function.
- (a) $f(x) = x^2 + 3$ (b) $f(x) = \sqrt{x}$ (c) $f(x) = \frac{1}{x}$

Answers: 1.(a) $(x+3)(x+2)$ (b) $(x-3)(x-2)$ (c) $(x+6)(x-1)$ (d) $(x+5)(x+1)$ (e) $(2x-3)(x+2)$
 (f) $(4x-3)(x+2)$ 2.(a) $x = \frac{1 \pm \sqrt{5}}{2}$ (b) $x = \frac{-3 \pm \sqrt{41}}{4}$ (c) No solutions. 3. Yes; although they have a different name, they have the same formula. 4.(a) 5 (b) -1 (c) 9
 5.



Introduction

The domain of a function is the collection of all acceptable input values. The range of a function is the collection of all acceptable output values, given the acceptable input values.

Astronomers use domain and range to determine and predict the paths of stars, asteroids, and falling debris. Astronomers determine functions that model the movement of these objects and then use the domain and range to find out the possible distances that they can travel. Engineers use domain and range to design highways and roads, and to report the strength of building materials. We implicitly use domain and range when looking at maps, graphs or charts, which are items that we look at quite often in our daily lives.

In mathematics textbooks, x is commonly used to represent the domain variable and y is commonly used to represent the range variable. At this stage of learning about mathematics, x and y are almost always considered to be real numbers, which means the domain of a function is either the set of all real numbers, or some real numbers. At higher levels of study, there are other possibilities for the elements of the domain and range of a function: they might be vectors, or matrices, or complex numbers, or many other possibilities. For our purposes, we shall always consider x and y to be real numbers.

To be precise, the specification of a function with just a formula is incomplete; the domain must also be specified. However, it's usual for mathematics textbooks to be lazy, and to omit a specific statement about the domain. Rather, there is an unstated agreement that the domain, unless otherwise specified, is the set of the real numbers that "makes sense." This leaves the task of determining the domain up to the reader! Once you've determined the domain, the range is then the corresponding set of real numbers.

In this module we'll explain this in detail and present examples to illustrate how to determine the domain and range of a function.

FOCUS QUESTION

To help you understand an important aspect of this lesson, focus your attention on this question, which will be answered towards the end of the lesson.

Determine the domain and range of the function $y = \frac{1}{x^2 + 2x + 2}$.

Keeping in mind that domain means the collection of all x -values that one can input into a function that make sense, let's explore the domain of some functions.

EXAMPLE 1

Determine the domain and range of the function $y = x + 3$.

SOLUTION

Let's begin by seeing what happens when we input various values for x . Having no idea which numbers might make sense, we'll just pick some relatively simple whole numbers to begin.

If we input $x = 3$ into this function, it makes sense because then $y = 3 + 3 = 6$.

Similarly, we can input $x = -2$ and this also makes sense because then $y = 3 + (-2) = 1$.

Continuing to try other values for x , we might soon guess that the domain of the function is all real values of x . But it's hard to feel confident about this conclusion, because we've only tried a few values for x . How can we be sure that there is no other value of x that doesn't make sense as an input value in the function?

Perhaps the following reasoning will be convincing: Think of the function in operational terms, as if you were going to determine the y -values by inputting the x -values into a calculator. Then we can describe the action of the function in words as follows: Input x , then add 3 to get the y -value. The question is, are there any values of x for which this operation would lead to nonsensical results? A little thought should convince you that all real values of x are acceptable, because it's possible to add 3 to any real number.

The conclusion is that the domain of this function is the set of all real numbers. This can be denoted in several equivalent ways; we'll use $\{x \in \mathbb{R}\}$ for now.

The range of the function is also the set of all real numbers: $\{y \in \mathbb{R}\}$. You can convince yourself of this by solving the function's formula for x , to obtain $x = y - 3$. Then ask yourself if there are any real values of y that do not have matching values of x . The answer is no: On the contrary, each real value of y has a corresponding value of x ; to determine the matching value of x , just subtract 3 from the y -value. This means that every value of y is produced as an output of the function. The conclusion is that the range of the function is all real values of y .

EXAMPLE 2

Determine the domain and range of the function $y = x^2 + 2$.

SOLUTION

If we try to input $x = -2$, $x = 0$, and $x = 2$ into the function we see that the corresponding function values are $y = 6$, $y = 2$, and $y = 6$, respectively. Exploring further, we would not discover any values of x that were troublesome. This might make us guess that the domain of the function is all real values of x . We could confirm this by reasoning as in the previous example: Thinking of the function in terms of operations, an input number x is squared, then 2 is added to the result. Is there any value of x for which we could not carry out these operations? No. Therefore, the domain of the function is all real values.

Now let's determine the range of the function. Notice that when we input $x = -2$, $x = 0$, and $x = 2$, the x^2 term is always greater than or equal to 0. This is an important property of squared numbers to remember! If the first term is always at least 0, and we always add 2 to this term, then the y -value will always be at least 2. Are all values of y that are greater than or equal to 2 realized as function values? (That is, do they have matching x -values?) Yes; to determine the matching x -values, just solve for x to obtain:

$$\begin{aligned}y &= x^2 + 2 \\x^2 &= y - 2 \\x &= \pm\sqrt{y - 2}\end{aligned}$$

Thus, there are two matching x -values for every y -value that is greater than 2. From this we can conclude that the range of the function is the set of all real numbers greater than or equal to 2. In symbols, the domain of the function is

$$\{y \in \mathbb{R} \mid y \geq 2\}$$

When determining the domain and range of a function, it is often helpful to ask, "Which values of the variable do *not* make sense?" We already began to use this reasoning in the previous examples. The following example illustrates this approach.

EXAMPLE 3

Determine the domain and range of the function $y = \frac{1}{x + 3}$.

SOLUTION

Let's try inputting some values x to see if the resulting function values make sense.

$$x = 3$$

$$y = \frac{1}{3+3}$$
$$= \frac{1}{6}$$

$$x = 0$$

$$y = \frac{1}{0+3}$$
$$= \frac{1}{3}$$

$$x = -3$$

$$y = \frac{1}{-3+3}$$
$$= \frac{1}{0} \quad (\text{which makes no sense})$$

We know that we cannot have 0 as a denominator so the input $x = -3$ is not allowed for this function. We can conclude that the domain does not include -3 . Are any other numbers excluded from the domain? We can find this out by seeing if any other values makes the denominator equal to 0.

$$x + 3 = 0$$
$$x = -3$$

We see that $x = -3$ is the only value that is excluded from the domain. Thus, the domain is the set of all real numbers except for $x = -3$. In symbols, the domain is $\{x \in \mathbb{R} \mid x \neq -3\}$.

Now, what about the range? Having a single number as our numerator allows us to quickly exclude a value from the range. Notice that because the numerator in the formula for y is 1, the value of the function cannot be 0. (The only way a fraction can be zero is if its numerator is zero and its denominator is not zero.) Are there any other values of the range that are excluded? To determine this, solve the formula for x :

$$y = \frac{1}{x+3}$$
$$x+3 = \frac{1}{y}$$
$$x = \frac{1}{y} - 3$$

Note that if we exclude $y = 0$, every other real y -value has a matching x -value. This means that the range is all real values of y except $y = 0$. In symbols, the range is $\{y \in \mathbb{R} \mid y \neq 0\}$.

Besides specifying a formula, it's also possible to describe a function simply by listing a set of ordered pairs. This is perhaps the preferred method when the graph of the function consists of just a few points. In this case, the domain is simply the list of x -values and the range is the list of

y -values. Both lists can be read off from the list of ordered pairs. Consider the following example.

EXAMPLE 4

Determine the domain and range for the following function.

$$\{(2, 3), (4, 6), (3, -1), (6, 6), (1, 3)\}$$

SOLUTION

The domain is $\{1, 2, 3, 4, 6\}$. The range is $\{-1, 3, 6\}$.

Note that there is no need to repeat values in the domain and range; that is, there is no need to write 3 or 6 twice for the range.

We can also read off the domain and range values when we are given a graph of a function.

EXAMPLE 5

Calculate the domain and range of the following graph.

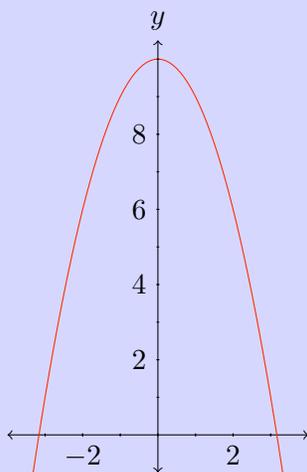


Figure 1: The graph of a function whose formula we do not know. (OK, we do know the formula, we're just not telling you what it is.)

SOLUTION

We have to make some assumptions to determine the domain and range here. And the solution will depend on the assumptions we make. So really we can't say for sure what the domain and range are. This goes back to our earlier comments about the laziness of mathematics textbooks; really the domain and range should be specified along with a formula or graph. But the real world is a messy place, not neat and tidy like the textbooks, so it is worthwhile practicing the skills needed for sorting out messes.

Let's assume that the graph is a parabola, which would mean that the function is quadratic. With this assumption, the vertex $(0, 10)$ is the maximum point on the graph, and the graph opens down and extends indefinitely in the x -direction.

Based on this assumption, we can see from the graph that the y -value is never larger than 10, making the range all real numbers less than or equal to 10. We can also see that x can be any real number. Therefore the domain is the set of all real numbers. We can input any value for x into this function and produce a meaningful y -value.

EXAMPLE 6

Determine the domain and range of the function $y = \frac{1}{x+5}$.

SOLUTION

Remember that $1/0$ does not exist, which means that the denominator $x+5$ cannot equal 0.

Which input for x would make the denominator 0? Setting the denominator equal to 0 and solving for x , we obtain:

$$\begin{aligned}x + 5 &= 0 \\x &= -5\end{aligned}$$

This means that x cannot equal -5 , but can be any other real number. Therefore, the domain is the set of all real numbers except $x = -5$. In other words, the domain is $\{x \in \mathbb{R} \mid x \neq -5\}$.

To determine the range, solve the formula for x in terms of y , and repeat the analysis:

$$\begin{aligned}y &= \frac{1}{x+5} \\x+5 &= \frac{1}{y} \\x &= \frac{1}{y} - 5\end{aligned}$$

There is clearly no matching x -value for $y = 0$; every other y -value has a matching x -value. Thus, the range of the function is the set of all real numbers except for $y = 0$; that is, the range is $\{y \in \mathbb{R} \mid y \neq 0\}$.

EXAMPLE 7

Determine the domain of the function $y = \frac{x}{x^2 - 7x + 10}$.

SOLUTION

We'll use the approach of first looking for input x -values for which the formula does not make sense. That is, we need to find out which x -value(s) make the denominator 0.

$$x^2 - 7x + 10 = 0$$

$$(x - 5)(x - 2) = 0$$

$$x = 5, x = 2$$

Therefore, the domain of the function is the set of all real numbers except $x = 2$ and $x = 5$.

To determine the range of this function is a little more involved; see the following investigation.

INVESTIGATION

Explore this on your own!

To determine the range of the function $y = \frac{x}{x^2 - 7x + 10}$, which was studied in the previous example, involves solving for x in terms of y , which is a fair bit of work. Instead of doing this, you could use a graphing calculator or software to plot the graph. This will give you some sense for what the range is. If you would like to do the work involved in determining the range, read on:

To solve for x in terms of y , make use of the quadratic formula. When you use the quadratic formula, the square root of an expression involving y results. Analyze this expression, noting that when the expression is negative, the square root of a negative quantity makes no sense. Thus, there is no matching x -value for the y -values that make the expression under the square root symbol negative.

After all this considerable amount of work, you should determine that the range of the function is the set of y -values that are less than or equal to about -1.481 (less than or equal to $\frac{-7 - 2\sqrt{10}}{9}$ to be exact) together with all y -values that are greater than or equal to about -0.075 (greater than or equal to $\frac{-7 + 2\sqrt{10}}{9}$ to be exact).

For functions that involve square roots, we need to remember that we can only have values that are greater than or equal to 0 under the square root. In other words, the square root of a negative number makes no sense, since we are working in the realm of real numbers. Consider the following example.

EXAMPLE 8

Determine the domain and range of the function $y = \sqrt{x}$.

SOLUTION

We know that the value under the square root has to be greater than or equal to 0; otherwise, there will be no output value. Therefore, the domain of this function is the set of all real numbers greater than or equal to 0.

What about the range of this function? We know that the square root of a function is always positive. Therefore, the y -values will also always be positive or equal to 0. The range for this function is therefore all real numbers greater than or equal to 0.

PRACTICE

(Answers below.)

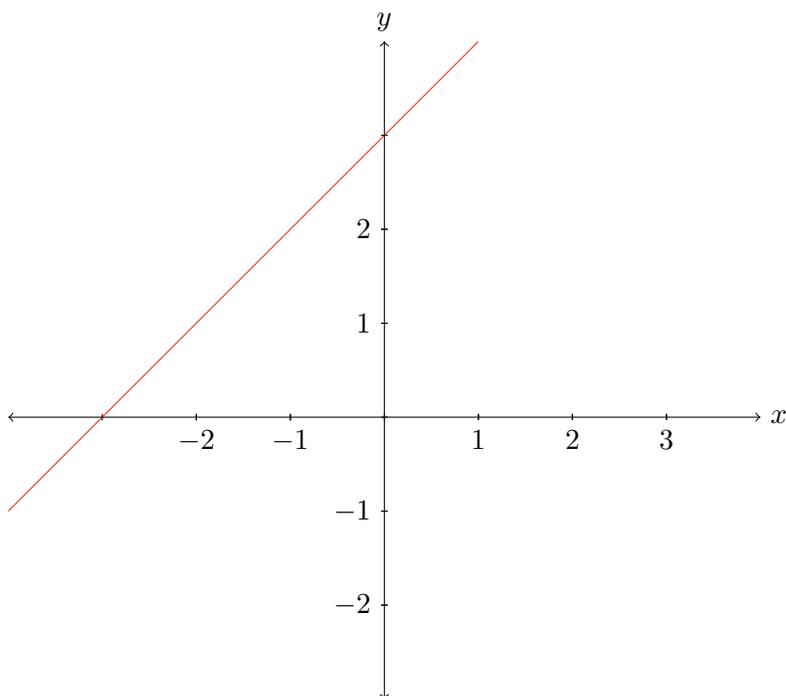
1. Determine the domain and range of the function.

$$\{(-2, 3), (-4, 6), (-6, 12), (-8, 24)\}$$

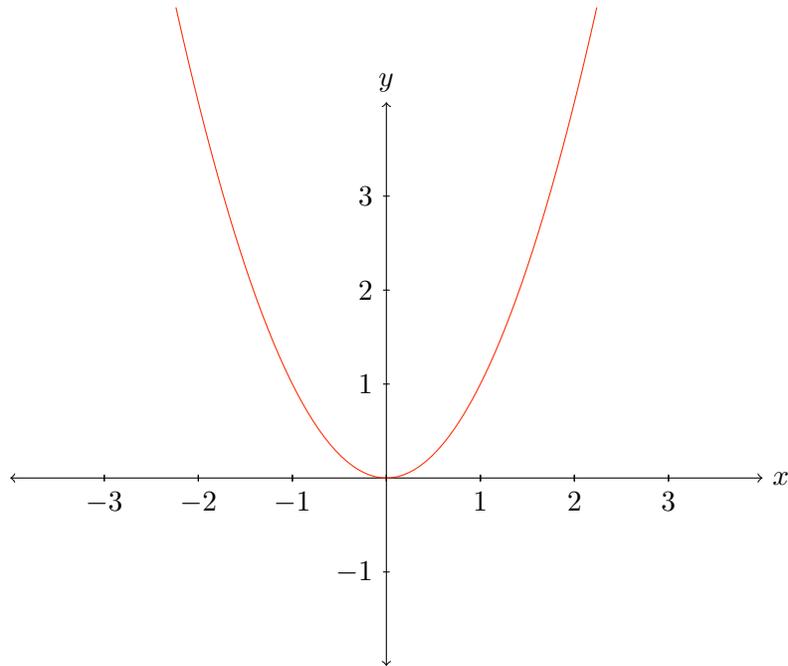
2. Determine the domain and range of the function.

$$\{(1, 3), (3, 5), (3, 7), (5, 3)\}$$

3. Determine the domain and range of the function by examining its graph.



4. Determine the domain and range of the function by examining its graph.



5. Determine the domain and range of the function $y = x - 2$.

6. Determine the domain and range of the function $y = x^2 + 2$.

7. Determine the domain and range of the function $y = \sqrt{x - 2}$.

8. Of the following three statements, which one is incorrect?

(a) For the function $y = 3x^2 + 6$, the range is the set of all real values of y that are greater than 6.

(b) For the function $y = \frac{x + 2}{(x + 3)(x - 4)}$, the domain is the set of all real values of x that are not equal to -3 or 4 .

(c) For the function $y = \sqrt{x}$, the domain is the set of all real values of x that are greater than or equal to 0.

9. Determine whether each statement is true or false.

- (a) The range of the function $y = 7 + 6x$ is the set of all real values of y that are greater than or equal to 7.
- (b) The domain of the function $y = x^2 + 13$ is the set of all real values of x , and the range is the set of all real numbers greater than or equal to 13.
- (c) The domain of the function $y = \sqrt{16 - x}$ is the set of all real values of x that are less than or equal to 16.

Answers: 1. Domain: $\{-8, -6, -4, -2\}$, Range: $\{3, 6, 12, 24\}$ 2. Domain: $\{1, 3, 5\}$, Range: $\{3, 5, 7\}$ 3. Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$ 4. Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R} \mid y \geq 0\}$ 5. Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$ 6. Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R} \mid y \geq 2\}$ 7. Domain: $\{x \in \mathbb{R} \mid x \geq 2\}$, Range: $\{y \in \mathbb{R} \mid y \geq 0\}$ 8. (a) is the incorrect statement 9.(a) False (b) True (c) True

RECAP OF FOCUS QUESTION

Recall the focus question, which was asked earlier in the lesson.

Determine the domain and range of the function $y = \frac{1}{x^2 + 2x + 2}$.

SOLUTION

The domain is all values of x that “make sense” in the formula for the function. The only values that make no sense are values of x for which the denominator is 0. Let’s see what these values of x are; set the denominator equal to 0 and solve for x using the quadratic formula:

$$\begin{aligned} 0 &= x^2 + 2x + 2 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} \\ &= \frac{-2 \pm \sqrt{-4}}{2} \end{aligned}$$

The presence of the negative value within the square root tells us that there are no real solutions to the equation. Therefore, there are no problematic x -values, and so the domain of the function is all real values of x .

For the range, notice that y is never 0; the only way this could happen is if the numerator of the right side of the equation could be equal to zero, and this can’t happen, because the numerator is 1. Also notice that y is never negative. You can see this as follows: If you plotted the quadratic expression $x^2 + 2x + 2$ by itself, you would see that it opens up. However, we determined in the previous paragraph that it never crosses the x -axis (it has no zeros), and so it is always positive. This means that y is also always positive, since it is 1 divided by a quantity that is positive.

There are at least two good ways to see that the maximum value of y is 1. One way is to determine the coordinates of the vertex of the denominator of the formula for y ; the vertex is at $(-1, 1)$. This tells us that the minimum value of the denominator is 1, which means that the maximum value of y is $1/1 = 1$. A second way to see this is to solve the function formula for x in terms of y :

$$\begin{aligned}
 y &= \frac{1}{x^2 + 2x + 2} \\
 (x^2 + 2x + 2)y &= 1 \\
 x^2 + 2x + 2 &= \frac{1}{y} \\
 x^2 + 2x + 2 - \frac{1}{y} &= 0 \\
 x &= \frac{-2 \pm \sqrt{2^2 - (4)(1)(2 - 1/y)}}{2(1)} \\
 x &= \frac{-2 \pm \sqrt{4 - (4)(2 - 1/y)}}{2} \\
 x &= \frac{-2 \pm \sqrt{4(1 - (2 - 1/y))}}{2} \\
 x &= \frac{-2 \pm 2\sqrt{1 - 2 + 1/y}}{2} \\
 x &= \frac{2(-1 \pm \sqrt{-1 + 1/y})}{2} \\
 x &= -1 \pm \sqrt{-1 + 1/y}
 \end{aligned}$$

We've already argued that y must be positive; the question now is which of the positive values of y have matching x -values. Because the square-root expression in the previous equation must be greater than or equal to 0, we can determine the allowed values of y as follows:

$$\begin{aligned}
 -1 + \frac{1}{y} &\geq 0 \\
 \frac{1}{y} &\geq 1 \\
 1 &\geq (1)(y) \quad (\text{multiplying both sides by } y) \\
 y &\leq 1
 \end{aligned}$$

Combining this with our previous reasoning yields the range: $\{y \in \mathbb{R} \mid 0 < y \leq 1\}$.

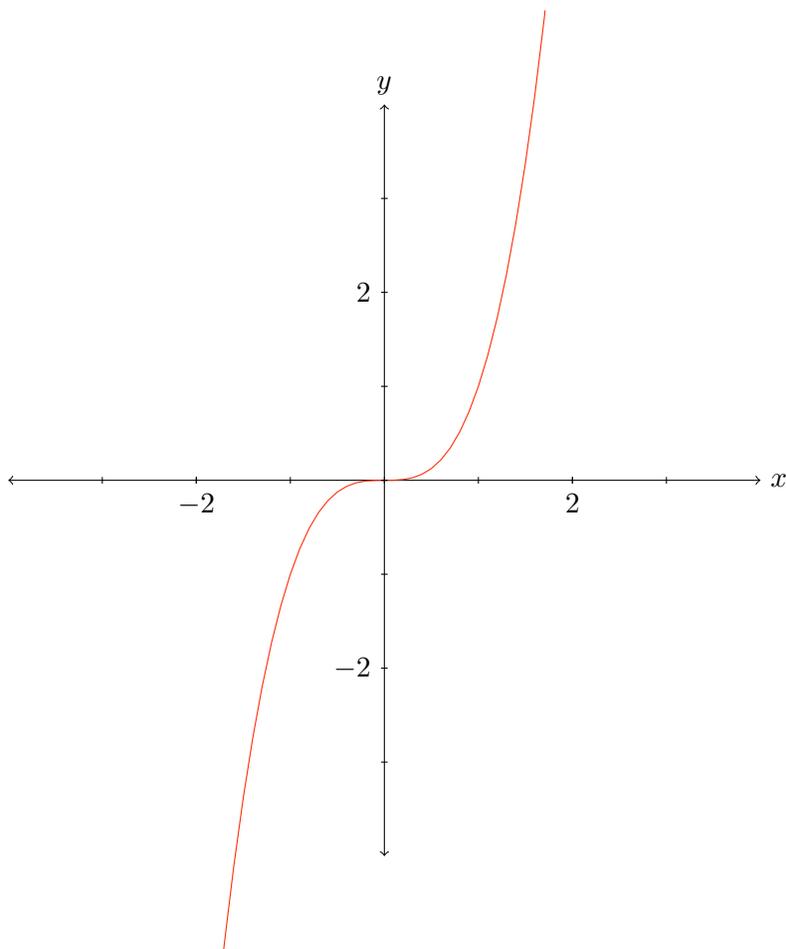
We have investigated the domain and range of some polynomial, rational, and square-root function. In later modules, you will learn how to apply domain and range to more complicated functions and situations.

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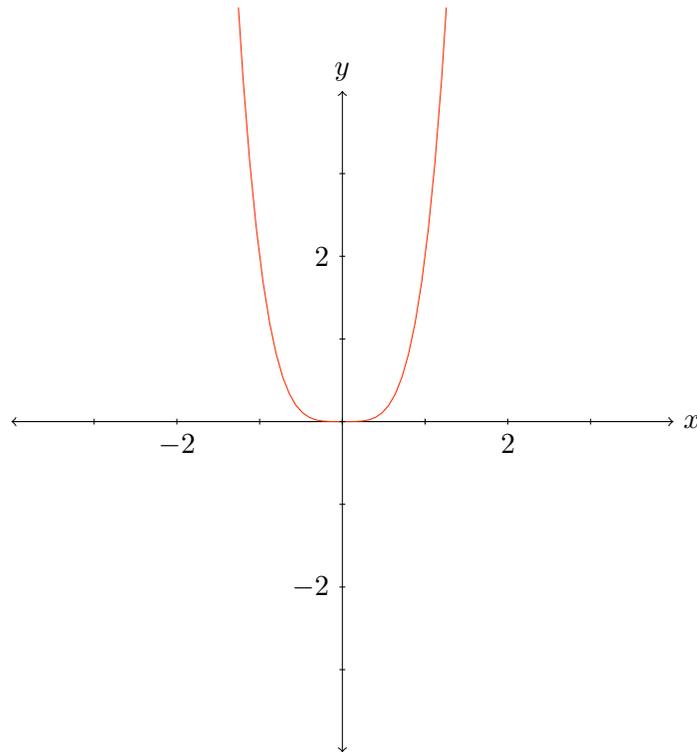
- What we did: Learned how to determine the domain and range of simple polynomial, rational, and square-root functions.
- Why we did it: The concepts of domain and range help us to understand functions and their graphs, and also help us to assess whether solutions to equations are acceptable.
- What's next: We can use the concepts of domain and range to help us graph functions. We can also expand our technique by learning to determine domain and range for a wider variety of functions.

EXERCISES

1. Determine the domain and range of the function $\{(-1, 1), (-1, 2), (-1, 3), (2, 5)\}$.
2. Determine the domain of the function.



3. Determine the range of the function.



4. Determine the domain and range of each function.

(a) $f(x) = 5x + 3$ (b) $g(x) = 6 + 4x^2$ (c) $y = 9x^3$

(d) $y = \sqrt{9 - x}$ (e) $y = -2(x + 1)^2 + 3$ (f) $y = \frac{1}{x}$

(g) $y = \frac{1}{3 - x}$ (h) $y = \frac{1}{\sqrt{x - 2}}$ (i) $y = x^2 - 4x$

(j) $g(x) = \sqrt{9 - 3x}$ (k) $y = \frac{x}{3x^2 + 8x + 4}$ (l) $y = x + \frac{1}{5x + 7}$

(m) $y = \frac{1}{x^2 + 6x + 8}$

5. Determine the equation of the line that has slope 3 and passes through the point $(2, 1)$. Then determine the domain and range of this function.

6. Determine the equation of the line that has a slope of -2 and passes through the point $(-3, 2)$. Then determine the domain and range of this function.

7. (a) Sarah has a summer job at a driving range. She receives 5 dollars every day that she comes into work plus 2 dollars for every ten golf balls that she picks up. Create a function that models the amount of money that Sarah could make in a day. Determine the domain and range of this function. What do the domain and range represent in this example?
- (b) Sarah is offered a job at another driving range. She is offered 7 dollars every day that she comes into work plus 3 dollars for every ten golf balls that she picks up with a maximum number of 300 golf balls that she can pick up each day. Determine the domain and range of this function. At which job would she have the possibility of making the most money?